1995

On Nonspecificity of fuzzy sets with continuous membership functions

George Klir
Bo Yuan

Follow this and additional works at: http://scholarworks.rit.edu/article

Recommended Citation

This Article is brought to you for free and open access by RIT Scholar Works. It has been accepted for inclusion in Articles by an authorized administrator of RIT Scholar Works. For more information, please contact ritscholarworks@rit.edu.
ON NONSPECIFICITY OF FUZZY SETS WITH CONTINUOUS MEMBERSHIP FUNCTIONS

George J. Klir and Bo Yuan
Department of Systems Science and Industrial Engineering
Binghamton University - SUNY
Binghamton, New York 13902-6000

ABSTRACT

A measure of nonspecificity is proposed for fuzzy sets with continuous membership functions defined, in general, on the n-dimensional Euclidean space \( \mathbb{R}^n \). The proposed measure is restricted to fuzzy sets that are convex. The measure is justified in terms of the usual axiomatic requirements.

1. INTRODUCTION

The well-established measure of uncertainty (and associated information) in classical set theory, referred to as the Hartley measure [3,10], was generalized to fuzzy set theory and possibility theory by Higashi and Klir [4]. For any nonempty fuzzy set \( F \) defined on a finite universal set \( X \), the generalized measure, \( U \), has the form

\[
U(F) = \frac{1}{h(F)} \int_0^{h(F)} \log_2 |\alpha F| d\alpha ,
\]

where \(|\alpha F|\) denotes the cardinality of the \( \alpha \)-cut of \( F \) and \( h(F) \in (0,1) \) is the height of \( F \). The uniqueness of this measure under well-justified axioms was proven by Klir and Mariano [8]. The type of uncertainty measured by function \( U \) is usually referred to in the literature as nonspecificity, and it is clearly distinguished from probabilistic uncertainty measured by the Shannon entropy [6,10]. However, the utility of this measure in dealing with problems involving fuzzy sets or possibility measures mirrors the well-known utility of the Shannon entropy in dealing with problems involving probability.

Although examples of applications of function \( U \) are known [6, 7], its restriction to finite sets has been somewhat unsettling since membership functions of fuzzy sets are often continuous functions defined on the set of real numbers. To alleviate this situation, we proposed a measure of nonspecificity, \( N \), for fuzzy sets with continuous membership function, which is a natural counterpart of function \( U \) [9]. Given a nonempty fuzzy set \( F \) defined on the set \( R \) of real numbers whose membership function is measurable and Lebesgue integrable, we define \( N \) by the formula

\[
N(F) = \frac{1}{h(F)} \int_0^{h(F)} \log_2 |\mu^{\alpha}(\alpha F)| d\alpha ,
\]

where \( \mu^{\alpha}(\alpha F) \) denotes the measure of the \( \alpha \)-cut, \( \alpha F \), defined by the Lebesgue integral of the characteristic function of \( \alpha F \). When \( \alpha F = [a, b] \), then \( \mu^{\alpha}(\alpha F) = b - a \). The natural logarithm in (2) is chosen for convenience since there is no reason to use the logarithm with any other base; in the finite case (1), the choice of the logarithm base 2 is motivated by the desire to employ bits as measurements units.

While the nonspecificity measure defined by (2) is adequate for fuzzy sets defined on \( R \), it is not directly applicable to fuzzy sets on \( \mathbb{R}^n \) when \( n > 1 \). The purpose of this paper is to introduce a generalization of this measure, which is applicable to convex fuzzy sets on \( \mathbb{R}^n \) for any finite \( n \).

Examining (2), we can see that \( N(F) \) is the weighted average of the Hartley measure,

\[
H(\alpha F) = \log_2 [1 + \mu^{\alpha}(\alpha F)] ,
\]

for the \( \alpha \)-cuts \( \alpha F \) of the given fuzzy set \( F \). The weights are normalized differences in the values of \( \alpha \) of successive \( \alpha \)-cuts of \( F \). Eq. (2) may thus be written as

\[
N(F) = \frac{1}{h(F)} \int_0^{h(F)} H(\alpha F) d\alpha .
\]

To extend the applicability of (4) to fuzzy sets on \( \mathbb{R}^n \) for any finite \( n \), we just need to appropriately generalize the Hartley measure, \( H(\alpha F) \), defined for each \( \alpha \)-cut of \( F \) by (3). Hence, the focus in this paper is on the formulation of the requirements for this generalized Hartley measure and on examining a particular function we propose as a generalization of (3) in terms of these requirements.

2. REQUIREMENTS FOR GENERALIZED HARTLEY MEASURE

Let \( X \) denote a universal set of concern that is convex and such that \( X \subseteq \mathbb{R}^n \) for some finite \( n \geq 1 \), and let \( A, B \)
denote arbitrary convex subsets of $X$. We may assume that $X$ is bounded in virtually all practical applications, but this assumption is not necessary for our mathematical treatment.

Let $N$ denote a function of the form

$$N : C \to \mathbb{R},$$

where $C$ is the family of all convex subsets of $X$, which for each $A \in C$ is expected to measure the nonspecificity of $A$. Then, $N$ must satisfy the following axiomatic requirements:

(N1) For each $A \in C$, $N(A) \in [0, \infty]$, where $N(A) = 0$ iff $A = \{x\}$ for some $x \in X$ (range requirement).

(N2) For all $A, B \in C$, if $A \subseteq B$, then $N(A) \leq N(B)$ (monotonicity).

(N3) For each $A \in C$, $N(A) \leq \sum_{i=1}^{n} N(A_i)$, where $A_i$ denotes the one-dimensional projection of $A$ to dimension $i$ in some coordinate system (subadditivity).

(N4) For each $A \in C$, such that $A = \bigotimes_{i=1}^{n} A_i$, where $A_i$ has the same meaning as in (N3), $N(A) = \sum_{i=1}^{n} N(A_i)$ (additivity).

(N5) Function $N$ does not change under isometric transformations of the coordinate system (coordinate invariance).

(N6) Function $N$ is a continuous function (continuity).

3. PROPOSED GENERALIZATION OF THE HARTLEY MEASURE

As a candidate for function $N$ characterized by the axiomatic requirements given in Sec. 2, whose purpose is to serve as a generalization of the Hartley measure, we propose the function defined for each $A \in C$ by the formula

$$N(A) = \min_{t \in T} \left[ \prod_{i=1}^{n} \left[ 1 + \mu(A_i) \right] + \mu(A) - \prod_{i=1}^{n} \mu(A_i) \right],$$

where $\mu$ denotes the Lebesgue measure and $T$ denotes the set of all transformation from one orthogonal coordinate system to another, and $A_i$ denotes the $i$th projection of $A$ within the coordinate system $t$. In the following, we examine this function from the standpoint of the axiomatic requirements (N1) – (N6).

It is evident that the proposed function is continuous, invariant with respect to isometric transformations of the coordinate system, and that it satisfies the required range. The monotonicity of the function follows directly from the corresponding monotonicity of the Lebesgue measure, and its subadditivity is demonstrated as follows: for any $A \in C$,

$$N(A) = \min_{t \in T} \left[ \prod_{i=1}^{n} \left[ 1 + \mu(A_i) \right] + \mu(A) - \prod_{i=1}^{n} \mu(A_i) \right]$$

$$\leq \min_{t \in T} \left[ \prod_{i=1}^{n} \left[ 1 + \mu(A_i) \right] \right]$$

$$= \min_{t \in T} \sum_{i=1}^{n} \ln \left[ 1 + \mu(A_i) \right]$$

$$= \sum_{i=1}^{n} N(A_i).$$

It remains to show that the proposed function is additive, to be fully justified as a general measure of nonspecificity of convex subsets of $\mathbb{R}^n$ for any finite $n \geq 1$.

4. ADDITIVITY OF THE PROPOSED FUNCTION

To prove that the proposed function $N$ is additive, we have to prove that

$$N(A) = \sum_{i=1}^{n} N(A_i)$$

for any $A \in C$ such that $A = \bigotimes_{i=1}^{n} A_i$. We have already shown that

$$N(A) \leq \sum_{i=1}^{n} N(A_i)$$

for any $A \in C$. Hence, it remains to prove that

$$N(A) \geq \sum_{i=1}^{n} N(A_i)$$

when $A = \bigotimes_{i=1}^{n} A_i$. This, in turn, amounts to proving that for any rotation of the set $A$,

$$\left[ \prod_{i=1}^{n} \left[ 1 + \mu(A_i) \right] + \mu(A) - \prod_{i=1}^{n} \mu(A_i) \right] \geq \prod_{i=1}^{n} \left[ 1 + \mu(A_i) \right].$$

26
At this time, we can only conjecture that this inequality holds. However, this conjecture has such a strong support, on several distinct grounds, that its validity is very likely. In the following, we present this support.

Generally, in the n-dimensional space \( \mathbb{R}^n \), any rotation can be represented by the orthogonal matrix

\[
I = \begin{bmatrix}
\cos \alpha_{11} & \cos \alpha_{12} & \cdots & \cos \alpha_{1n} \\
\cos \alpha_{21} & \cos \alpha_{22} & \cdots & \cos \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\cos \alpha_{n1} & \cos \alpha_{n2} & \cdots & \cos \alpha_{nn}
\end{bmatrix},
\]

where the parameters \( \alpha_{ij} \) satisfy the following properties:

1. \( \sum_{j=1}^{n} \cos^2 \alpha_{ij} = 1, \quad \forall j \in \mathbb{N}_n = \{1, 2, \ldots, n\}, \) and
   \( \sum_{i=1}^{n} \cos^2 \alpha_{ij} = 1, \quad \forall i \in \mathbb{N}_n. \)

2. \( \sum_{k=1}^{n} \cos \alpha_{ik} \cos \alpha_{jk} = 0, \quad \forall i, j \in \mathbb{N}_n, \) and
   \( \sum_{k=1}^{n} \cos \alpha_{ki} \cos \alpha_{kj} = 0, \quad \forall i, j \in \mathbb{N}_n. \)

For each given rotation defined by matrix \( I \), any arbitrary point

\[
x = (x_1, \ldots, x_n)^T
\]

in \( \mathbb{R}^n \) is transformed to the point

\[
x' = (x'_1, \ldots, x'_n)^T
\]

by the matrix equation

\[
x' = Ix.
\]

That is,

\[
x'_1 = x_1 \cos \alpha_{11} + x_2 \cos \alpha_{12} + \cdots + x_n \cos \alpha_{1n}
\]

\[
x'_2 = x_1 \cos \alpha_{21} + x_2 \cos \alpha_{22} + \cdots + x_n \cos \alpha_{2n}
\]

\[
\vdots
\]

\[
x'_n = x_1 \cos \alpha_{n1} + x_2 \cos \alpha_{n2} + \cdots + x_n \cos \alpha_{nn}.
\]

Let us consider, without any loss of generality, that

\[
A = \bigotimes_{i=1}^{n} [0, a_i]
\]

for some \( a_i \in \mathbb{R}, i \in \mathbb{N}_n \). Then, the \( i \)th projection of this set subjected to rotation defined by matrix \( I \) is the set

\[
A_i = \{ x_i' | x' = Ix, \quad \forall x \in A \},
\]

for any \( i \in \mathbb{N}_n \). The Lebesgue measure of the projection is

\[
\mu(A_i) = \max \{|x'_i - y'_i| \mid \forall x, y \in A\}.
\]

That is,

\[
\mu(A_i) = \max \left( \sum_{j=1}^{n} (x_j - y_j) \cos \alpha_{ij} \right) \mid \forall x, y \in A\}
\]

for any \( i \in \mathbb{N}_n \). Since this maximum must be reached by two vertices of the set \( A \), the Lebesgue measure of the projection can be rewritten by

\[
\mu(A_i) = \sum_{k=1}^{n} a_k |\cos \alpha_{ik}|
\]

for any \( i \in \mathbb{N}_n \).

Using the last formula, let us examine some aspects of the proposed function pertaining to its additivity. First, let us present the following two basic properties.

(a) \( \sum_{i=1}^{n} \mu(A_i) \geq \sum_{i=1}^{n} a_i \).

This is because

\[
\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_k |\cos \alpha_{ik}|
\]

\[
= \sum_{k=1}^{n} \sum_{i=1}^{n} a_k |\cos \alpha_{ik}|
\]

\[
\geq \sum_{k=1}^{n} \sum_{i=1}^{n} a_k \cos^2 \alpha_{ik}
\]

\[
= \sum_{k=1}^{n} a_k \sum_{i=1}^{n} \cos^2 \alpha_{ik} = \sum_{k=1}^{n} a_k.
\]

(b) \( \prod_{i=1}^{n} \mu(A_i) \geq \prod_{i=1}^{n} a_i \).

This is because of the fact that the Lebesgue measure of the set is less than or equal to the Lebesgue measure of the Cartesian product of one-dimensional projections.
The Case of Sets With Equal Edge Lengths

Let $a_i = a$ for all $i \in \mathbb{N}_n$. Then, we have

$$\mu(A_i) = \sum_{k=1}^{n} a_k \cos \alpha_{ik} = a \sum_{k=1}^{n} \cos \alpha_{ik}$$

and, consequently,

$$\left( \prod_{i=1}^{n} [1 + \mu(A_i)] - \mu(A) \right) \geq \left( \prod_{i=1}^{n} [1 + \mu(A_i)] - \mu(A) \right)$$

Hence, the additivity holds.

The Two-Dimensional Case

Let set $A$ be a rectangle in the standard coordinate system that is shown in Fig. 1. Since $A = [0, a_1] \times [0, a_2]$ in this system, we have

$$\prod_{i=1}^{2} (1 + \mu(A_i)) + \mu(A) - \prod_{i=1}^{2} \mu(A_i) = (1+a_1)(1+a_2).$$

Now, we prove that this is the minimum for all rotations. In the two-dimension space, any rotation can be represented by the matrix

$$I = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$  

Fig. 1 illustrates a rotated rectangle $A$ and its projections. It is easy to show that

$$\mu(A_1) = a_1 |\cos \theta| + a_2 |\sin \theta|$$

$$\mu(A_2) = a_1 |\sin \theta| + a_2 |\cos \theta|$$

Then, under the new coordinate system,

$$\prod_{i=1}^{2} (1 + \mu(A_i)) + \mu(A) - \prod_{i=1}^{2} \mu(A_i)$$

$$= 1 + a_1 |\cos \theta| + a_2 |\sin \theta| + a_1 |\sin \theta| + a_2 |\cos \theta| +$$

$$\left( a_1 |\cos \theta| + a_2 |\sin \theta|)(a_1 |\sin \theta| + a_2 |\cos \theta|) + a_1 a_2$$

$$\geq 1 + a_1 |\cos^2 \theta + a_2 |\sin^2 \theta| + a_1 |\sin\theta|^2 + a_2 |\cos \theta|^2 + (a_1 |\cos^2 \theta + a_2 |\sin^2 \theta|)^2$$

$$\geq (1+a_1)(1+a_2).$$

Therefore, in the two-dimension space, the measure is additive.

The Three-Dimensional Case

To prove (6) in the three dimensional space, we only need to prove that for any rotation of a set $A \in \mathbb{C}$

$$\mu(A_1) \mu(A_2) + \mu(A_1) \mu(A_3) + \mu(A_2) \mu(A_3)$$

$$\geq a_1 a_2 + a_1 a_3 + a_2 a_3.$$  

Since the Cartesian product of the projections includes the original set as a subset and both of them are cubes, the area of the surface of the Cartesian product, which is twice of the left hand side of the above inequality, is greater than or equal to the area of the surface of the set $A$, which is twice of the right hand side. Therefore, the inequality holds.

The General $n$-Dimensional Case

As mentioned previously, we have not yet been able to prove that the proposed function is additive for sets with unequal edges when $n > 3$. However, we tested the additivity of the function on the computer for thousands of randomly generated examples for each $n = 4, 5, ..., 10$. In all these examples, additivity of the function has been verified. We also have not been able to conceive of any way to construct a counterexample by which our conjecture could be falsified.

5. CONCLUSIONS

A generalized Hartley measure of uncertainty, which is applicable to convex subsets of $\mathbb{R}^n$, is introduced in this paper. Relevant properties of the introduced measure are examined, and it is shown how to utilize the measure for measuring nonspecificity of fuzzy sets with continuous membership functions. We have also shown that the proposed measure satisfies all essential axiomatic requirements, with the possible exception of additivity. Although we have
not yet been able to prove that the measure is additive, we
gathered strong evidence supporting its additivity; neverthe-
less, a general proof of its additivity remains an open prob-
lem.

We conceive of many applications of the proposed
measure, which we intend to explore in the near future.Infor-
mation-preserving conversions of fuzzy sets to crisp sets,
which can be utilized, for example, in fuzzy database sys-
tems or fuzzy information retrieval systems, is one of these
applications [7]. Another application to be explored is the
use of the introduced measure for dealing with the problem
of choosing appropriate defuzzification method in fuzzy con-
trol systems. In addition, the measure seems to be useful in
the important and broad area of approximate reasoning [1, 2]. The possibility of measuring uncertainty of IF-THEN
rules was already examined by Kacprzyk [5]. We intend to
evaluate the known approximate reasoning schemes accord-
ing to the minimum specificity principle [1] and propose a
new inference schemes based on the introduced measure of
nonspecificity. The measure can also be applied in the area of
statistical modeling with imprecise probabilities, where it
enables us to calculate nonspecificities associated with con-
 vex sets of probability distributions.

6. ACKNOWLEDGEMENTS

Research sponsored by Rome Laboratory, Electronic
Systems Center, Air Force Materiel Command, USF, under
grant number RL F30602-94-1-001. The US Government is
authorized to reproduce and distribute reprints for govern-
mental purposes not withstanding any copyright notation
thereon. The views and conclusions contained in this docu-
ment are those of the authors and should not be interpreted as
necessarily representing the official policies or endorsement,
either expressed or implied, of Rome Laboratory or the US
Government.

We would also like to acknowledge a creative contri-
bution by Jiri Fridrich regarding the proof of additivity for n
= 3.

7. REFERENCES

[1] Dubois, D. and H. Prade [1991], “Fuzzy sets in approx-
imate reasoning, Part 1: Inference with possibility distribu-

[2] Dubois, D. and H. Prade [1991], “Fuzzy sets in approx-
imate reasoning, Part 2: Logical approaches.” Fuzzy Sets and
Systems, 40(1), pp. 202-244.


tainty and information based on possibility distribution.”

IF-THEN rules.” Intern. J. of Approximate Reasoning, 11(1),
pp. 29-53.


of possibilistic measure of uncertainty and information.”

Logic: Theory and Applications. Prentice Hall PTR, Upper
Saddle River, NJ.

New York.