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Approximate Solutions of Systems of Fuzzy Relation Equations

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Abstract- In this paper, we generalize our previous results regarding approximate solutions of fuzzy relation equations [10] to systems of fuzzy relation equations. By employing the equality index proposed by Gottwald [3], we introduce a goodness measure of the performance of approximate solutions and derive a lower bound and an upper bound of solvability of systems of fuzzy relation equations. We also show that our results generalize those presented for one-dimensional fuzzy relation equations in [2].

I. INTRODUCTION

Fuzzy relation equations, which are now recognized as one of the most important subjects of fuzzy set theory [2], were first investigated by Sanchez [7]. This subject is not only mathematically rich, but it also offers a broad spectrum of significant applications in areas such as fuzzy control, approximate reasoning, knowledge engineering, system identification, medical diagnosis, and many others.

Let \( \mathcal{P}(X) \) denote the fuzzy power set of \( X \) (the set of all functions from \( X \) to \([0,1]\)). Then, the basic problem of solving fuzzy relation equations can be stated as follows: given fuzzy relations \( Q \in \mathcal{P}(X \times U) \) and \( S \in \mathcal{P}(W \times V) \), determine fuzzy relations \( R \in \mathcal{P}(X \times V) \) that satisfy the equation

\[
Q \circ_i R = S,
\]

where \( i \) is a t-norm [8], which is usually required to be continuous, and \( \circ_i \) denotes the sup-i composition of fuzzy relations. Eq.(1) may also be written as

\[
\sup_{u \in U} (Q(w,u),R(u,v)) = S(w,v)
\]

for all \( w \in W \), \( v \in V \).

Currently, the theory of fuzzy relation equations relies, by and large, on the assumption that the solution set is not empty. However, this is often not the case in practical applications. Hence, we need a broader theory that would allow us to determine an adequate approximate solution when no real solution exists.

Existing methods for obtaining approximate solutions of fuzzy relation equations are reviewed by Di Nola et al. [1]. One of the methods, ostensibly the first one described in the literature, is based upon a modified Newton method for solving fuzzy relation equations numerically. According to this method, which was proposed by Pedrycz [5], approximate solutions are those fuzzy relations \( R \) for which \( Q \circ_i R \) has a minimal distance from \( S \). Unfortunately, the structure of the approximate solution set obtained by this method is not clear. Another method, proposed by Gottwald and Pedrycz [4], is based on a solvability index, which, in turn, is based on the equality index introduced for fuzzy sets by Gottwald [3]; the larger the solvability index, the easier to solve the equation. According to the method, the given equation is appropriately modified to increase the solvability index until an equation is obtained that is solvable. As we argued elsewhere [10], this method is rather inefficient and leads usually to a trivial solution.

Another definition of approximate solutions of a given system of fuzzy relation equations was proposed by Wu [9]. According to this definition, one part of fuzzy relation equations corresponds to modus ponens, while the other part corresponds to modus tollens in approximate reasoning. Contrary to Pedrycz's definition of approximate solutions in terms of a distance, Wu's definition is based upon the lattice structure of the fuzzy power set. Employing Wu's definition, we generalize in this paper our previous work regarding approximate solutions of fuzzy relation equations [10].
II. PRELIMINARIES

Given a continuous t-norm \(i\) [8], we define another function, \(\alpha : [0,1]^2 \to [0,1]\), by
\[
\alpha (a,b) = \sup \{x \mid i(a,x) \leq b\}, \quad \forall a,b \in [0,1].
\] (3)

Due to the continuity of the t-norm \(i\), \(\alpha\) is well defined by (3). Function \(\alpha\), which is often called a residual operator, is used to define a fuzzy implication (referred to as R-implication) in approximate reasoning. It plays also a fundamental role in solving fuzzy relation equations as well as in determining approximate solutions of fuzzy relation equations that have no real solutions.

The following are some well-established basic properties of operator \(\alpha\) [2,9]:

\begin{enumerate}
  \item \(\min(a,b)=i(a,\alpha(a,b))\leq \alpha(a,a(b))\);
  \item \(i(a,b)\leq \alpha(a,c)\leq \alpha(a,b)\);
  \item \(\alpha(i(a,b),c)=\alpha(a,\alpha(b,c))\);
  \item \(\alpha(a,b)=\alpha(\alpha(a,c),\alpha(b,c))\);
  \item \(\alpha(\inf_{\alpha(a,b)}=\sup_{\alpha(a,b)}\alpha(a,b))\);
  \item \(\alpha(\sup_{\alpha(a,b)}=\inf_{\alpha(a,b)}\alpha(a,b))\);
  \item \(\alpha(b,\sup_{\alpha(a,b)}=\sup_{\alpha(a,b)}\alpha(b,a))\);
  \item \(\alpha(\inf_{\alpha(a,b)}=\sup_{\alpha(a,b)}\alpha(a,b))\);
  \item \(\alpha(a,\alpha(i(a,b),b))=\alpha(a,b)\);
\end{enumerate}

where \(a,b,c,a,b \in [0,1]\) and \(j \in J\) is an index set. Note that the equalities in (v) and (vii) hold when \(J\) is finite.

By employing the residual operator, we can define inf-\(\alpha\) composition of two fuzzy relations. Let \(P \in \mathcal{F}(U \times W), Q \in \mathcal{F}(W \times V)\). Then the inf-\(\alpha\) composition of \(P\) and \(Q\), denoted by \(P \circ_{\alpha} Q\), is defined by
\[
(P \circ_{\alpha} Q)(u,v)=\alpha(Q(u,w),Q(w,v)),
\] (4)
for any \(u \in U, v \in V\).

Now we introduce one of the most important theorems pertaining to fuzzy relation equations [1].

**Theorem 1.** The solution set of the fuzzy relation equation (1) for \(R\) is non empty if and only if the fuzzy relation \(Q^1 \circ_{\alpha} S^1\) is the greatest solution of the relation for \(R\).

III. APPROXIMATE SOLUTIONS

In this section, we introduce the definition of approximate solutions for a fuzzy relation equation whose solution set is empty. The way to obtain an approximate solution is to modify the original equation as far as the solution set of the modified equation is non empty. The following definition is a variant of the definition suggested by Wu [9].

**Definition 1.** A fuzzy relation \(\tilde{R}\) is called an approximate solution of (1) if the following requirements are satisfied:

\begin{enumerate}
  \item there exist \(Q' \supseteq Q\) and \(S' \subseteq S\) such that \(Q' \circ \tilde{R} = S'\);
  \item if there exist \(Q'' \subseteq Q\), \(S'' \subseteq S\) and \(Q'' \circ R'' = S''\), then \(Q'' = Q'\) and \(S'' = S'\).
\end{enumerate}

Note that if \(\tilde{R}\) is an approximate solution of (1), then it is an exact solution associated with \(Q'\) and \(S'\). Requirement (i) means that we pursue the approximate solutions of (1) by making \(Q'\) larger and \(S'\) smaller until the solution set is non empty; requirement (ii) means that \(Q'\) and \(S'\) are the closest relations to \(Q\) and \(S\), respectively, for which a solution exists. Clearly, if the solution set of (1) is non empty, then approximate solutions are exact solutions of (1).
EXAMPLE 1. Consider the equation
\[
\begin{bmatrix}
.1 & .2 \\
.3 & .4
\end{bmatrix} \circ R = \begin{bmatrix}
.5 \\
.6
\end{bmatrix},
\]
where \(i\) is the algebraic product. Since, according to Theorem 1,
\[
\begin{bmatrix}
.1 & .2 \\
.3 & .4
\end{bmatrix} \circ \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
.1 & .2 \\
.3 & .4
\end{bmatrix} \circ \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
2 \\
.6
\end{bmatrix},
\]
this equation has no solution. To pursue an approximate solution of the equation, let us reduce \(S = \begin{bmatrix}
.5 \\
.6
\end{bmatrix}\) to \(S' = \begin{bmatrix}
.2 \\
.4
\end{bmatrix}\).

Then \(Q^{-1} \circ S = \begin{bmatrix}
1 \\
1
\end{bmatrix}\) and \(\begin{bmatrix}
.1 & .2 \\
.3 & .4
\end{bmatrix} \circ \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
2 \\
.4
\end{bmatrix}\). Assume that there exists \(S'' = \begin{bmatrix}
s_1 \\
s_2
\end{bmatrix}\) such that \(S' \subseteq S'' \subseteq S\) and the equation \(\begin{bmatrix}
.1 & .2 \\
.3 & .4
\end{bmatrix} \circ R'' = S''\) has a solution for \(R''\), say \(R'' = \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}\). Then,
\[
\max(1r_1, 2r_2) = s_1,
\max(3r_1, 4r_2) = s_2.
\]

This equation can be satisfied only when \(s_1 \leq 2\) and \(s_2 \leq 4\). That means \(S'' \subseteq S'\); hence, \(S'' = S\). This implies, by Definition 1, that \(R = \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}\) is an approximate solution of the original equation. Furthermore, we can easily see that \(\begin{bmatrix}
1 \\
1
\end{bmatrix}\) for any \(a \in [0, 1]\) is also an approximate solution of the equation. That is, approximate solutions are not unique.

Let us pursue now approximate solutions of the given equation by increasing \(Q\) to \(Q' = \begin{bmatrix}
.1 & .5 \\
.3 & .6
\end{bmatrix}\). Then \(\begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}\) is a solution of the equation \(Q' \circ R = S\). If there is \(Q'' = \begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{bmatrix}\) such that \(Q'' \subseteq Q' \circ Q''\) and there exists a solution for \(R'' = \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}\) such that \(Q'' \circ R'' = \begin{bmatrix}
5 \\
6
\end{bmatrix}\), then this means that \(q_{11} = 1, q_{21} = 3, q_{12} \in [2.5], q_{22} \in [4.6]\). Then,
\[
\begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{bmatrix} \circ \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix} = \begin{bmatrix}
5 \\
6
\end{bmatrix},
\]
which represents the simple equations
\[
\max(1r_1, q_{12} r_2) = 5,
\max(3r_1, q_{22} r_2) = 6.
\]
That is, \(q_{12} r_2 = 5\) and \(q_{22} r_2 = 6\), which implies \(q_{12} \geq 5\) and \(q_{22} \geq 6\). Hence \(Q'' \supseteq Q'\), and therefore \(Q'' = Q'\). Again, by Definition 1, \(\bar{R}\) is an approximate solution of the original equation.

We can see from the above example that not only approximate solutions of fuzzy relation equations are not unique, but also the modified relation \(Q'\) and \(S'\) that facilitate the approximate solutions are not unique. The following theorem guarantees that approximate solutions always exist.

**Theorem 2.** \(\bar{R} = Q^{-1} \circ S\) is the greatest approximate solution of (1).

**Proof.** First, we need to verify that \(\bar{R}\) satisfies requirements (i) and (ii) of Definition 1.

Let \(Q' = Q, S' = Q \circ (Q^{-1} \circ S)\). Then, clearly, \(Q' \circ \bar{R} = S'\) and \(S' = Q \circ (Q^{-1} \circ S) \subseteq S\). Thus (i) of Definition 1 is satisfied.

Assume now that there exist \(R''\) and \(S''\) such that \(S' \subseteq S'' \subseteq S\) and \(Q'' \circ R'' = S''\). Then \(Q^{-1} \circ S''\) is the greatest solution for \(R''\). Thus \(Q \circ (Q^{-1} \circ S'') = S''\). Since \(S' \subseteq S'' \subseteq S\), we have \(Q^{-1} \circ S' \subseteq Q^{-1} \circ S'' \subseteq Q^{-1} \circ S\). Moreover,
($Q^{-1} o_{S'} = Q^{-1} o_{S} [Q o_{(Q^{-1} o_{S})} S] = Q^{-1} o_{S}$). Therefore, $Q^{-1} o_{S} S' \subseteq Q^{-1} o_{S} S' \subseteq Q^{-1} o_{S} S = Q^{-1} o_{S} S'$. Hence, (ii) of Definition 1 is verified.

It remains to show that $\hat{R}$ is the greatest approximate solution of (1). Assume that $\hat{R}'$ is another approximate solution, i.e., there exist $Q'$ and $S'$ such that $Q' \subseteq Q$, $S' \subseteq S$ and $Q' o_{\hat{R}'} = S'$. Then, we have $\hat{R}' \subseteq Q^{-1} o_{S'} S' \subseteq Q^{-1} o_{S} S = \hat{R}$. 

Note that if $\hat{R}$ is an approximate solution associated with $Q'$ and $S'$, then any solutions of the equation $Q' o_{\hat{R}'} = S'$ for $\hat{R}'$ will be approximate solutions of the original equation.

IV. GOODNESS MEASURE OF APPROXIMATE SOLUTIONS

In this section, we introduce a goodness measure to evaluate approximate solutions by employing the equality index introduced by Gottwald [3] and Gottwald and Pedrycz [4]. Let $A, B \in \mathfrak{F}(U)$. The index of $A \subseteq B$, denoted by $\|A \subseteq B\|$, which indicates the degree of truth of the statement “$A$ is included in $B$,” is defined by

$$\|A \subseteq B\| = \inf_{u \in U} (A(u), B(u)).$$

This index has the following properties:

(a) $\|A \subseteq B\| \in [0, 1]$;
(b) $\|A \subseteq B\| = 1$ iff $A \subseteq B$;
(c) $C \subseteq B$ implies $\|A \subseteq B\| \geq \|A \subseteq C\|$ and $\|C \subseteq A\| \geq \|B \subseteq A\|$.

Now, the equality index of $A$ and $B$ is defined by

$$\|A = B\| = \min \{\|A \subseteq B\|, \|B \subseteq A\|\} = \inf_{u \in U} \min \{\omega_{A}(A(u), B(u)), \omega_{B}(B(u), A(u))\}.$$

Obviously, $A = B$ if and only if $\|A - B\| = 1$.

Employing the equality index, we introduce now a goodness index of an approximate solution.

**Definition 2.** Let $\hat{R}$ be an approximate solution for $R$ in (1). The goodness of $\hat{R}$, denoted by $G(\hat{R})$, is defined by

$$G(\hat{R}) = \|Q o_{\hat{R}} = S\|.$$

Note that the goodness of $\hat{R}$ is expressed in terms of the degree of equality of the left-hand side of (1), when $\hat{R}$ is employed in the composition, and its right-hand side. According to $G(\hat{R})$, the greater the equality degree, the better $\hat{R}$ approximates the equation. Moreover, $G(\hat{R}) = 1$ if and only if $\hat{R}$ is an exact solution of (1).

Employing the goodness index, the following theorem shows that the greatest approximate solution $\hat{R}$ is the best.

**Theorem 3.** The fuzzy relation $\hat{R} = Q^{-1} o_{S}$ is the best approximate solution of fuzzy relation equation (1) in terms of goodness index defined by (7).

**Proof.** To prove this theorem, we have to show that $G(R') \leq G(\hat{R})$ for any other approximate solution $R'$ of (1). Since $Q \subseteq \hat{R} \subseteq S$, we have $\|Q \subseteq \hat{R} \subseteq S\| = 1$ and

$$G(\hat{R}) = \|Q o_{\hat{R}} S\| = \min \{\|Q o_{\hat{R}} S\|, \|S \subseteq Q o_{\hat{R}}\|\} = \|S \subseteq Q o_{\hat{R}}\|.$$

Assume that $R'$ is any approximate solution of (1). Then, $R' \subseteq \hat{R}$, by Theorem 2. Therefore,

$$Q o_{R'} S \subseteq Q o_{\hat{R}} S,$$

Thus, $\|Q o_{R'} S\| = 1$, i.e., $G(R') = \|S \subseteq Q o_{\hat{R}}\|$. It follows from property (c) of $\|A \subseteq B\|$ and (8), that

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The solvability index was first introduced by Gottwald and Pedrycz [4]. The index characterizes the ease in solving fuzzy relation equations of the general form (1). The solvability index of (1), \( \delta \), is defined by

\[
\delta = \sup\{ Q \circ R \mid R \in \mathcal{F}(U \times V) \}. \tag{9}
\]

It is obvious that \( \delta \in [0,1] \) and, moreover, if (1) has an exact solution, then \( \delta = 1 \). However, the inverse implication remains, in general, an open question. When \( \delta < 1 \), the given equation has no exact solutions. The smaller the value of \( \delta \), the more difficult it is to solve the equation approximately. Since \( Q^{-1} \circ S \in \mathcal{F}(U \times V) \), we have \( \delta \geq \| Q^{-1} \circ S \| = G(\tilde{R}) \). Thus, the goodness index of the greatest approximate solution of (1) is a lower bound of the solvability index \( \delta \). The following theorem establishes an upper bound of \( \delta \).

**Theorem 4.** Let \( \delta \) be the solvability index of fuzzy relation equation (1). Then

\[
G(\tilde{R}) \leq \delta = \inf_{w \in W} \{ \sup_{v \in V} S(w,v), \sup_{w \in W} Q(w,u) \}. \tag{10}
\]

**Proof.** For any \( R \in \mathcal{F}(U \times V) \),

\[
\| Q \circ R \| = \min\{ Q \circ R \mid R \in \mathcal{S} \} \leq \| S \circ Q \circ R \|
\]

\[
= \inf_{w \in W} \inf_{v \in V} \{ S(w,v), (Q \circ R)(w,v) \}
\]

\[
= \inf_{w \in W} \inf_{v \in V} \{ S(w,v), \sup_{u \in U} Q(w,u) \}
\]

\[
= \inf_{w \in W} \inf_{v \in V} \{ S(w,v), \sup_{u \in U} Q(w,u) \}
\]

\[
= \inf_{w \in W} \inf_{v \in V} \{ S(w,v), \sup_{u \in U} Q(w,u) \}.
\]

Therefore,

\[
\delta = \sup\{ Q \circ R = S \mid R \in \mathcal{F}(U \times V) \} \leq \inf_{w \in W} \{ \sup_{v \in V} S(w,v), \sup_{u \in U} Q(w,u) \}.
\]

We can easily show that this theorem is a generalization of a result obtained by Gottwald and Pedrycz [4] for simple one-dimensional fuzzy relation equations.

Let \( A \in \mathcal{F}(U) \), \( B \in \mathcal{F}(V) \), and

\[
A \circ R = B. \tag{11}
\]

We can convert (11) into our form. Let \( W = \{ w \} \). Then \( \| W \| = 1 \). Setting

\[
Q(w,u) = A(u), \quad S(w,v) = B(v),
\]

for any \( u \in U \) and \( v \in V \), (11) becomes

\[
Q \circ R = S, \tag{12}
\]

where \( Q \in \mathcal{F}(W \times U) \), \( S \in \mathcal{F}(W \times V) \).

**Corollary.** The solvability index of (11) is

\[
\delta = \omega_0(\text{hgt}(B), \text{hgt}(A)).
\]

**Proof.** It is obvious that the solvability index of (11) is equal to the solvability index of (12). Then, by Theorem 4, we have

\[
\delta \leq \inf_{w \in W} \{ \sup_{v \in V} S(w,v), \sup_{u \in U} Q(w,u) \}
\]

\[
= \omega_0(\text{hgt}(B), \text{hgt}(A)).
\]
This corollary shows that Theorem 4 is a generalization of the result obtained by Gottwald and Pedrycz [4].

VI. CONCLUSION
In this paper, we first reintroduce the definition of approximate solutions that was originally suggested by Wu [9]. Then, we prove that $Q_{1}S_{1}$ is the greatest approximate solution of fuzzy relation equation (1). Secondly, by employing the equality index introduced by Gottwald [3], we define an appropriate goodness index for approximate solutions and show that $Q_{1}S_{1}$ is also the best approximate solution in terms of this index. Finally, we establish an upper bound and a lower bound of the solvability index of fuzzy relation equation (1). We also show that the bounds collapse into a single formula obtained by Gottwald and Pedrycz [4] for the one-dimensional case.

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