8-31-2015

Some Computational and Theoretical Problems for Ramsey Numbers

Stanislaw P. Radziszowski
Rochester Institute of Technology, spr@cs.rit.edu

Follow this and additional works at: http://scholarworks.rit.edu/other

Recommended Citation

This Presentation is brought to you for free and open access by RIT Scholar Works. It has been accepted for inclusion in Presentations and other scholarship by an authorized administrator of RIT Scholar Works. For more information, please contact ritscholarworks@rit.edu.
Some Computational and Theoretical Problems for Ramsey Numbers
bounds, constructions and connectivity

Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology, NY, USA

Third Gdańsk Workshop on Graph Theory
September 16, 2015
Ramsey Numbers

- \( R(G, H) = n \) iff minimal \( n \) such that in any 2-coloring of the edges of \( K_n \) there is a monochromatic \( G \) in the first color or a monochromatic \( H \) in the second color.

- 2-colorings \( \cong \) graphs, \( R(m, n) = R(K_m, K_n) \)

- Generalizes to \( k \) colors, \( R(G_1, \cdots, G_k) \)

- Theorem (Ramsey 1930): Ramsey numbers exist
Unavoidable classics

\[ R(3, 3) = 6 \]

\[ R(3, 5) = 14 \] [GRS'90]
Asymptotics

diagonal cases

- **Bounds** (Erdős 1947, Spencer 1975; Conlon 2010)
  \[
  \frac{\sqrt{2}}{e} 2^{n/2} n < R(n, n) < R(n + 1, n + 1) \leq \left( \frac{2n}{n} \right)^n n^{-c \log n / \log \log n}
  \]

- **Conjecture** (Erdős 1947, $100)
  \[\lim_{n \to \infty} R(n, n)^{1/n} \text{ exists.}\]
  If it exists, it is between \(\sqrt{2}\) and 4 ($250 for value).

- **Theorem** (Chung-Grinstead 1983)
  \[L = \lim_{k \to \infty} R_k(3)^{1/k} \text{ exists.}\]
  \[3.199 < L, \text{ (Fredricksen-Sweet 2000, Xie-Exoo-R 2004)}\]
Asymptotics
Ramsey numbers avoiding $K_3$

- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound

\[ R(3, n) = \Theta \left( \frac{n^2}{\log n} \right) \]

- Bohman 2009, triangle-free process
  Bohman-Keevash 2013
  Fiz Pontiveros-Griffiths-Morris 2013

Shearer 1983 (upper bound)

\[ \left( \frac{1}{4} + o(1) \right) \frac{n^2}{\log n} \leq R(3, n) \leq (1 + o(1)) \frac{n^2}{\log n} \]
Off-Diagonal Cases

upper bounds

- Erdős-Szekeres 1935 (implicit)
  \[ R(m, n) \leq \binom{m + n - 2}{m - 1} \]

- Ajtai-Komlós-Szemerédi 1980, Graham-Rödl 1981 for fixed \( n \geq 3 \) and large \( m \)
  \[ R(m, n) \leq c^n m^{n-1} / (\log m)^{n-2} \]
Off-Diagonal Cases
fixed small avoided $K_m$

- Bohman triangle-free process - 2009
  probabilistic lower bound

\[ R(4, n) = \Omega\left(\frac{n^{5/2}}{\log^2 n}\right) \]

\[ R(4, n) = O\left(\frac{n^3}{\log^2 n}\right) \]

- Kostochka, Pudlák, Rödl - 2010
  constructive lower bounds

\[ R(4, n) = \Omega\left(\frac{n^8}{5}\right), \quad R(5, n) = \Omega\left(\frac{n^{5/3}}{}\right), \quad R(6, n) = \Omega\left(n^2\right) \]

(vs. probabilistic $5/2, 6/2, 7/2$ with $/\log$s for $4, 5, 6$)
Clebsch \((3, 6; 16)\)-graph on \(GF(2^4)\)

\((x, y) \in E \text{ iff } x - y = \alpha^3\)

Alfred Clebsch (1833-1872)
#vertices / #graphs
no exhaustive searches beyond 13 vertices

<table>
<thead>
<tr>
<th>#vertices</th>
<th>#graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>34</td>
</tr>
<tr>
<td>6</td>
<td>156</td>
</tr>
<tr>
<td>7</td>
<td>1044</td>
</tr>
<tr>
<td>8</td>
<td>12346</td>
</tr>
<tr>
<td>9</td>
<td>274668</td>
</tr>
<tr>
<td>10</td>
<td>12005168</td>
</tr>
<tr>
<td>11</td>
<td>1018997864</td>
</tr>
<tr>
<td>12</td>
<td>165091172592</td>
</tr>
<tr>
<td>13</td>
<td>50502031367952 $\approx 5 \times 10^{13}$</td>
</tr>
<tr>
<td></td>
<td>-------too many to process--------</td>
</tr>
<tr>
<td>14</td>
<td>29054155657235488 $\approx 3 \times 10^{16}$</td>
</tr>
<tr>
<td>15</td>
<td>31426485969804308768</td>
</tr>
<tr>
<td>16</td>
<td>64001015704527557894928</td>
</tr>
<tr>
<td>17</td>
<td>245935864153532932683719776</td>
</tr>
<tr>
<td>18</td>
<td>$\approx 2 \times 10^{30}$</td>
</tr>
</tbody>
</table>
From the deductive mathematics point of view most of these results are not theorems, being only descriptions of several millions of particular observations. However, I hope that they are even more important than the formal deductions from the formal axioms, providing new points of view on difficult problems where no other approaches are that efficient.
Values and bounds on $R(k, l)$

two colors, avoiding $K_k, K_l$

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>14</td>
<td>18</td>
<td>23</td>
<td>28</td>
<td>36</td>
<td>40</td>
<td>47</td>
<td>52</td>
<td>59</td>
<td>66</td>
<td>73</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>59</td>
<td>73</td>
<td>92</td>
<td>100</td>
<td>128</td>
<td>136</td>
<td>146</td>
<td>155</td>
<td>155</td>
</tr>
<tr>
<td>5</td>
<td>43</td>
<td>58</td>
<td>80</td>
<td>101</td>
<td>133</td>
<td>148</td>
<td>171</td>
<td>194</td>
<td>218</td>
<td>242</td>
<td>267</td>
<td>267</td>
<td>267</td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>115</td>
<td>134</td>
<td>175</td>
<td>185</td>
<td>253</td>
<td>263</td>
<td>317</td>
<td>401</td>
<td>5033</td>
<td>6911</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>205</td>
<td>217</td>
<td>242</td>
<td>289</td>
<td>405</td>
<td>417</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>282</td>
<td>317</td>
<td>3583</td>
<td>6090</td>
<td>10630</td>
<td>16944</td>
<td>817</td>
<td>27485</td>
<td>41525</td>
<td>63609</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>565</td>
<td>581</td>
<td>12677</td>
<td>22325</td>
<td>38832</td>
<td>64864</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>798</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[ElJC survey *Small Ramsey Numbers*, revision #14, 2014, with numerous updates]
Bounds on $R(3, k) − R(3, k − 1)$

Erdős and Sós, 1980, asked about

$$3 \leq \Delta_k = R(3, k) − R(3, k − 1) \leq k,$$

$$\Delta_k \xrightarrow{k} \infty \quad \Delta_k/k \xrightarrow{k} 0 ?$$

Look at $R(K_3, K_k - e)$ relative to $R(K_3, K_k) = R(3, k)$

$$\Delta_k = (R(K_3, K_k) - R(K_3, K_k - e)) + (R(K_3, K_k - e) - R(K_3, K_{k-1}))$$
\( \Delta_k = R(3, k) - R(3, k - 1) \)

It is known that

\[
\left( \frac{1}{4} + o(1) \right) k^2 / \log k \leq R(3, k) \leq (1 + o(1)) k^2 / \log k
\]

All we have for \( K_k \) and \( K_k - e \) is

(1) \( 3 \leq R(3, K_k) - R(3, K_{k-1}) \leq k \), easy old bounds,
(2) \( R(3, K_{k-1}) \leq R(3, K_k - e) \leq R(3, K_k) \), trivial bounds,
(3) \( 4 \leq R(3, K_{k+1}) - R(3, K_k - e) \) (Zhu-Xu-R 2015).

**Problem.** Improve over any of the inequalities in (1), (2) or (3), or their combination as (3) combines parts of (1) and (2).
Δₜ ≥ 3

All graphs Δ-free

\[ Iₙ \leq s-1 \]

\[ d \leq s-1 \]

\[ \text{added 3 nodes} \]

\[ Iₙ \leq s-2 \]

\[ \text{no } Kₙ \]

\[ Iₙ \leq s-1 \]
$R(3, s + t - 1) \geq R(3, s + 1) + R(3, t + 1) - c_{st}$
$$R(3, K_{k+1}) - R(3, K_k - e) \geq 4$$

$$k = s + 1$$

---

**All graphs Δ-free**

- **Graph G**
  - Nodes: $x_1$, $x_2$, $x_{12}$
  - Edges: $m$

- **Graph H**
  - Nodes: $y_1$, $y_2$
  - Edges: $n=8$

**Additivity**

- $s \geq 4$

**Isolation**

- $t = 3$
  - $r = m + 4$

**Table**

<table>
<thead>
<tr>
<th></th>
<th>$X_{12}$</th>
<th>$X_{T2}$</th>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max IS in $F$ is $s+1$</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td>$s-1$</td>
<td>$s-1$</td>
<td>$s-2$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td>$2$</td>
<td>$2$</td>
<td>$3$</td>
<td></td>
</tr>
</tbody>
</table>
Known bounds on $R(3, K_s)$ and $R(3, K_s - e)$

$s_s = K_s - e$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$R(3, J_s)$</th>
<th>$R(3, K_s)$</th>
<th>$\Delta_s$</th>
<th>$s$</th>
<th>$R(3, J_s)$</th>
<th>$R(3, K_s)$</th>
<th>$\Delta_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>6</td>
<td>3</td>
<td>10</td>
<td>37</td>
<td>40–42</td>
<td>4–6</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td>11</td>
<td>42–45</td>
<td>47–50</td>
<td>5–10</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>14</td>
<td>5</td>
<td>12</td>
<td>47–53</td>
<td>52–59</td>
<td>3–12</td>
</tr>
<tr>
<td>6</td>
<td>17</td>
<td>18</td>
<td>4</td>
<td>13</td>
<td>55–62</td>
<td>59–68</td>
<td>3–13</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
<td>23</td>
<td>5</td>
<td>14</td>
<td>59–71</td>
<td>66–77</td>
<td>3–14</td>
</tr>
<tr>
<td>8</td>
<td>25</td>
<td>28</td>
<td>5</td>
<td>15</td>
<td>69–80</td>
<td>73–87</td>
<td>3–15</td>
</tr>
<tr>
<td>9</td>
<td>31</td>
<td>36</td>
<td>8</td>
<td>16</td>
<td>73–91</td>
<td>82–98</td>
<td>3–16</td>
</tr>
</tbody>
</table>

Table: $R(3, J_s)$ and $R(3, K_s)$, for $s \leq 16$ (Goedgebeur-R 2014).
Corollaries from constructions

**General**, but perhaps not that strong

- \( R(3, s + t) \geq R(3, s + 1) + R(3, t + 1) - 3 \),
- \( R(3, s + t - 1) \geq R(3, K_{s+1} - e) + R(3, K_{t+1} - e) - 5 \).

**Applications**, but quite interesting

- For \( s \geq 3 \) and \( m = R(3, s + 1) - 1 \), if there exists a \((3, s + 1; m)\)-graph which is not bicritical, then \( \Delta_{s+2} \geq 4 \),
- \( R(3, s + 1) \geq R(3, K_s - e) + 4 \).
Conjecture 1
and 1/2 of Erdős-Sós problem

Observe that
\[ R(3, s + k) - R(3, s - 1) = \sum_{i=0}^{k} \Delta_{s+i}. \]

We know that
\[ \Delta_s \geq 3, \Delta_s + \Delta_{s+1} \geq 7, \Delta_s + \Delta_{s+1} + \Delta_{s+2} \geq 11. \]

Conjecture 1
There exists \( d \geq 2 \) such that \( \Delta_s - \Delta_{s+1} \leq d \) for all \( s \geq 2 \).

Theorem
If Conjecture 1 is true, then \( \lim_{s \to \infty} \Delta_s / s = 0. \)
Conjecture 2
possibly easier to prove

Conjecture 2
There exists integer $k$ such that

$$\lim_{s \to \infty} \sum_{i=0}^{k} \Delta s+i = \infty.$$ 

Growth of $\Delta s$ gives also insights on

- connectivity of Ramsey-critical graphs, Beveridge-Pikhurko 2008, Xu-Shao-R 2011
- hamiltonicity of Ramsey-critical graphs, Xu-Shao-R 2011
- chromatic gap, Gyárfás-Sebő-Trotignon 2012
- Shannon capacity of graphs, Xu-R 2013
Construction

Chung-Cleve-Dagum 1993

Construction of $H \in \mathcal{R}(3, 9; 30)$ using $G = C_5 \in \mathcal{R}(3, 3; 5)$

$R(3, k) = \Omega(k^{\log 6/\log 4}) \approx \Omega(k^{1.29})$

Explicit $\Omega(k^{3/2})$ construction:
Alon 1994, Codenotti-Pudlák-Resta 2000
### History of bounds on $R(5, 5)$

<table>
<thead>
<tr>
<th>year</th>
<th>reference</th>
<th>lower</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1965</td>
<td>Abbott</td>
<td>38</td>
<td>quadratic residues in $\mathbb{Z}_{37}$</td>
</tr>
<tr>
<td>1965</td>
<td>Kalbfleisch</td>
<td>59</td>
<td>pointer to a future paper</td>
</tr>
<tr>
<td>1967</td>
<td>Giraud</td>
<td>58</td>
<td>LP</td>
</tr>
<tr>
<td>1968</td>
<td>Walker</td>
<td>57</td>
<td>LP</td>
</tr>
<tr>
<td>1971</td>
<td>Walker</td>
<td>55</td>
<td>LP</td>
</tr>
<tr>
<td>1973</td>
<td>Irving</td>
<td>42</td>
<td>sum-free sets</td>
</tr>
<tr>
<td>1989</td>
<td>Exoo</td>
<td>43</td>
<td>simulated annealing</td>
</tr>
<tr>
<td>1992</td>
<td>McKay-R</td>
<td>53</td>
<td>(4, 4)-graph enumeration, LP</td>
</tr>
<tr>
<td>1994</td>
<td>McKay-R</td>
<td>52</td>
<td>more details, LP</td>
</tr>
<tr>
<td>1995</td>
<td>McKay-R</td>
<td>50</td>
<td>implication of $R(4, 5) = 25$</td>
</tr>
<tr>
<td>1997</td>
<td>McKay-R</td>
<td>49</td>
<td>long computations</td>
</tr>
<tr>
<td>2014</td>
<td>McKay-Lieby</td>
<td></td>
<td>study of $(5, 5; 42)$-graphs</td>
</tr>
</tbody>
</table>
43 ≤ R(5, 5) ≤ 49

**Conjecture.** McKay-R 1997

\[ R(5, 5) = 43, \text{ and the number of } (5, 5; 42)-\text{graphs is precisely 656}. \]

The known \((5, 5; 42)\)-graphs have properties:

- # edges ranges from 423 to 438 (midpoint 430-431)
- mindeg 19, maxdeg 22
- no more than 2 symmetries
  - 232 graphs have only an involution
  - 424 graphs have trivial group
- none is almost regular
Define the distance between two graphs on $n$ vertices to be $k$ if their largest common induced subgraph has $n - k$ vertices.

- Very large computational effort to find distances between known $(5, 5; 42)$-graphs.
- Known $(5, 5; 42)$-graphs form one large cluster.
- McKay and Lieby report that any new $(5, 5; 42)$-graph $H$ would have to be in distance at least 6 from every graph in the set of 656 known $(5, 5; 42)$-graphs.
What to do next?

Challenges

Theoretical
- better explicit constructive lower bounds for $R(3, k)$
- improve bounds for $\Delta_k$, or a similar local difference
- generalize above beyond triangle-free graphs

Computational - improve any of the following
- $29 \leq R(C_5, K_8) \leq 33$
- $42 \leq R(3, K_{11} - e) \leq 45$
- Lower bounds for other larger parameters
- Other small puzzling $R(G, H)$, survey SRN 2014
What not to compute
infeasible without a breakthrough

Each of the following needs a new insight

▶ $R(3, 10) \leq 41$?
▶ $R(4, 6) \leq 40$?
▶ $R(5, 5) \leq 48$?

Seems hard to improve any of the corresponding lower bounds.
In 1980, Paul Erdős wrote

*Faudree, Schelp, Rousseau and I needed recently a lemma stating*

\[
\lim_{n \to \infty} \frac{r(n+1, n) - r(n, n)}{n} = \infty.
\]

*We could prove it without much difficulty, but could not prove that \( r(n+1, n) - r(n, n) \) increases faster than any polynomial of \( n \). We of course expect*

\[
\lim_{n \to \infty} \frac{r(n+1, n)}{r(n, n)} = C_{1/2}^{1/2},
\]

*where \( C = \lim_{n \to \infty} r(n, n)^{1/n} \).*

*The best known bound for \( r(n+1, n) - r(n, n) \) is \( \Omega(n) \).*
Theorem.

If $k \geq 5$ and $s \geq 3$, then the connectivity of any Ramsey-critical $(k, s)$-graph is no less than $k$.
(improves by 1 Beveridge-Pikhurko 2008)

Theorem.

If $k \geq s - 1 \geq 1$ and $k \geq 3$, except $(k, s) = (3, 2)$, then any Ramsey-critical $(k, s)$-graph is Hamiltonian.

In particular, for $k \geq 3$, all diagonal Ramsey-critical $(k, k)$-graphs are Hamiltonian.
\[ R_r(3) = R(3, 3, \ldots, 3) \]

- Much work on Schur numbers \( s(r) \) via sum-free partitions and cyclic colorings
  \[ s(r) > 89^{r/4-c\log r} > 3.07^r \text{ [except small } r] \]
  Abbott+ 1965+

- \( s(r) + 2 \leq R_r(3) \)
  \[ s(r) = 1, 4, 13, 44, \geq 160, \geq 536 \]

- \( R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3 \)
  Chung 1973

- The limit \( L = \lim_{r \to \infty} R_r(3)^{\frac{1}{r}} \) exists
  Chung-Grinstead 1983

  \[ (2s(r) + 1)^{\frac{1}{r}} = c_r \approx_{(r=6)} 3.199 < L \]
\( R(3, 3, 3) = 17 \)

two Kalbfleisch \((3, 3, 3; 16)\)-colorings, each color is a Clebsch graph
Four colors - $R_4(3)$

$51 \leq R(3,3,3,3) \leq 62$

<table>
<thead>
<tr>
<th>year</th>
<th>reference</th>
<th>lower</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1955</td>
<td>Greenwood, Gleason</td>
<td>42</td>
<td>66</td>
</tr>
<tr>
<td>1967</td>
<td>false rumors</td>
<td>[66]</td>
<td></td>
</tr>
<tr>
<td>1971</td>
<td>Golomb, Baumert</td>
<td>46</td>
<td></td>
</tr>
<tr>
<td>1973</td>
<td>Whitehead</td>
<td>50</td>
<td>65</td>
</tr>
<tr>
<td>1973</td>
<td>Chung, Porter</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>1974</td>
<td>Folkman</td>
<td></td>
<td>65</td>
</tr>
<tr>
<td>1995</td>
<td>Sánchez-Flores</td>
<td></td>
<td>64</td>
</tr>
<tr>
<td>1995</td>
<td>Kramer (no computer)</td>
<td></td>
<td>62</td>
</tr>
<tr>
<td>2004</td>
<td>Fettes-Kramer-R (computer)</td>
<td></td>
<td>62</td>
</tr>
</tbody>
</table>

History of bounds on $R_4(3)$ [from FKR 2004]
\[ 30 \leq R(3, 3, 4) \leq 31 \]

**Theorem.** Kalbfleisch 1965

\[ 30 \leq R(3, 3, 4) \]

**Theorem.** Piwakowski-R 2001

\[ R(3, 3, 4) = 31 \text{ if and only if there exists a } (3, 3, 4; 30)\text{-coloring } C \text{ such that every edge in the third color has at least one endpoint } x \text{ with } \deg_{C[3]}(x) = 13. \text{ Furthermore, } C \text{ has at least 25 vertices } v \text{ such that } \deg_{C[1]}(v) = \deg_{C[2]}(v) = 8 \text{ and } \deg_{C[3]}(v) = 13. \]
$R(3, 3, 4)$

**Note, March 2015.**
Codish, Frank, Itzhakov, Miller, arXiv posting

BEE (Metodi-Codish 2013), Ben-Gurion University Equi-propagation Encoder, a compiler to encode finite domain constraint problems to CNF.

Very significant progress of work towards the Ramsey number $R(3, 3, 4)$. Namely, they apply a SAT-solver to prove that if any $(3, 3, 4; 30)$-coloring exists, then it must be 8-regular in the first two colors and 13-regular in the third.

Furthermore, they anticipate that full analysis of all such colorings will be completed, and thus the exact value of $R(3, 3, 4)$ will be known soon.
Theorem. Alon and Rödl 2005

\[ R(3, 3, k) = \Theta(k^3 \text{poly-log } k). \]

More general:
Avoid triangles in the first \( r - 1 \) colors and \( K_k \) in color \( r \), then we have

\[ R(3, \ldots, 3, k) = \Theta(k^r \text{poly-log } k). \]

A nice, open, intriguing, feasible to solve case (Exoo 1991, Piwakowski 1997)

\[ 28 \leq R_3(K_4 - e) \leq 30 \]
What to do next?
theoretically and what to compute

Find new smart lower bound constructions

Explore relations between limits and Shannon capacity

Three colors
- improve $28 \leq R_3(K_4 - e) \leq 30$
- improve $45 \leq R(3, 3, 5) \leq 57$
- finish off $30 \leq R(3, 3, 4) \leq 31$

Four colors
- understand why heuristics don’t find $51 \leq R_4(3)$
- improve on $R_4(3) \leq 62$
Papers to look at

- SPR, revision #14 of the survey paper *Small Ramsey Numbers* at the *ElJC*, January 2014.


Thanks for listening!