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Mechanism Analysis Using Implicit Constraints

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MECHANISMS ANALYSIS USING IMPLICIT CONSTRAINTS

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ABSTRACT

This paper introduces an approach to kinematic and dynamic mechanisms analysis where one or more joints are modeled using joint component relative displacements that approximate real joint behavior. This approach allows for the simultaneous nonrecursive solution for both mechanism kinematic parameters and selected dynamic joint reaction forces. Also, for closed loop mechanisms, the approach eliminates the need for forming explicit loop closure constraint equations, so that the dynamic equations of motion, derived using either the Newtonian or Lagrangian method, have a simplified unconstrained form. The key element underlying the approach is the formation of axioms for the standard mechanism joint types that describe the form of the joint reaction force and/or moment in terms of a virtual (or real) displacement between the joint components.

Keywords: flexible joints, nonideal joints, mechanism dynamics, joint forces, loop closure constraints

1 INTRODUCTION

Connected multi-body systems (hereafter referred to as mechanisms for brevity) consist of closed loops (linkages), open loops (manipulator arms), or combinations of both (Figure 1). Joints of various standard configurations connect the bodies thereby constraining the bodies to move relative to each other in only certain allowed ways. The organized categorizing of mechanisms by their joints (and by some accounts the beginning of modern mechanisms science) dates back to Reuleaux [1] in 1875. While much work was done examining the geometric properties of simple mechanisms in the time both before and after Reuleaux, it was the emergence of digital computers that encouraged researchers to develop algebraic (versus graphical) techniques for the analysis and design of mechanisms. With the publication of Hartenberg and Denavit's text [2] in 1964, transformation matrices became popular (other than for the simplest planar devices) to describe the general relative orientation of adjacent joints (of a standard type) in (particularly spatial) mechanisms. The product of the transformation matrices can be used to form loop equations that can be used to analyze any mechanisms with standard joints. Uicker [3] is considered the first to publically take these joint coordinate transformation matrices to form a general purpose computer program. In the same timeframe using a different type of transformation matrix, Chace, starting with the theory outlined in [4], and ultimately in collaboration with Orlandea [5] created a comprehensive computer program, (that in its final form is called) ADAMS, to analyze the dynamics of spatial mechanisms based on the constraint relationships provided by joint transformation matrices. ADAMS continues to be the standard for comparing new approaches to certain classes of multi-body dynamic analysis problems [6].

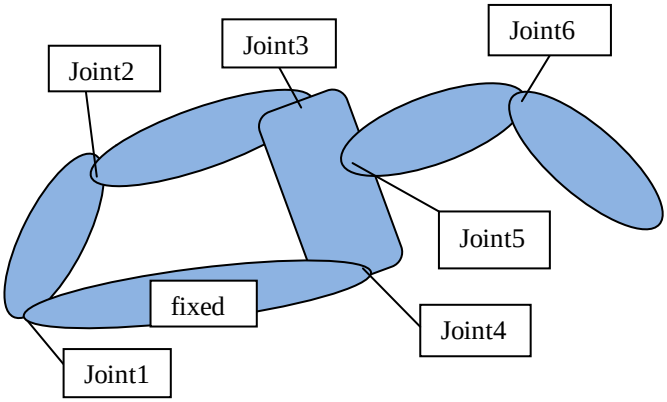

Figure 1 Representative Combined Closed and Open Loop Mechanism

Typical closed and open loop mechanisms are holonomic in the classical sense [7,8,9] in that knowing the generalized coordinates of the system will uniquely (at least to within a finite set of possibilities for closed loop mechanisms) determine the absolute position of the mechanism components. However for closed loop mechanisms it is common practice [4,6,8] to differentiate the kinematic closure constraint position equations and use either the first or second derivative equations as the constraint equations for the system, thus treating the system as if it were nonholonomic. Depending on the analyst's choice of system coordinates and approach to forming the dynamic equilibrium equations, using the differentiated kinematic constraint equations can have advantages over using the non-differentiated constraint equations. The standard form for handling constrained dynamical systems using Lagrange multipliers [7,9] requires constraint equations in the assumed form of being first order differential equations that are linear in the first derivatives, but with the possibility of the coefficients being nonlinear functions of the generalized coordinates and time.

2 NONIDEAL JOINTS

Mechanism joints are traditionally considered to be "ideal" joints where they are infinitely stiff with respect to certain directions of possible connected body motion and offer zero resistance to relative motion in other defined directions. The constraints that joints, and the equations derived from them, provide are what distinguishes one mechanism from another, since the unconstrained equations of motion are the same for bodies with the same inertia properties. When the behavior of real joints is considered [10,11] in the form of clearances, friction and elastic behavior, the motion of the connected bodies – and particularly the forces within the bodies -- can deviate to a varying degree from that for ideal joint behavior. However the consideration of nonideal joint behavior has required additional modeling of considerable complexity. Thus, unless it is known a priori that mechanism joint clearance, friction and/or elasticity are significant effects for the problem at hand, they are not considered (due to additional model complexity and computational expense) when mechanisms are analyzed in practice, and ideal joint behavior is therefore customarily assumed.

It is the basis of the Implicit Constraint Approach to take an axiomatic approach to building up the dynamic equations of motion for mechanisms, where joints are represented not by kinematic constraints describing the relative motion they provide between connected bodies, but rather by joint forces of a simplified form that approximates real joint behavior. The assumed form of the joint forces is not in the form of unknown scalar variables, as is found in traditional Newtonian or Lagrangian multiplier analysis [7,9], but in the form of a kinematic expression that does not introduce new mechanisms variables/coordinates. By taking this approach the equations of motion for the system can be formed directly using equilibrium or energy principles without the need for additional constraint equations that enforce linkage closure. In many cases this can lead to a simplified more stable set of system dynamic equations than those resulting from the assumption of ideal joint behavior. Furthermore the approach lends itself to including real joint effects as an intrinsic part of the initial mechanism dynamic analysis, rather than as a computationally expensive additional analysis step. This would encourage nonideal joint behavior to be routinely included in the standard practice for mechanism analysis.

3 THE IMPLICIT CONSTRAINT APPROACH

The Implicit Constraint Approach (ICA), does not require the explicit formation of kinematic loop closure constraint equations (or their equivalent) for closed loop mechanisms. Instead, one or more joints in the loop are opened up and the joint kinematic constraints are replaced by constraint forces that are assumed to have a specific kinematic form which depends on the type of joints being replaced. The system equations of motion are then written using either an unconstrained Newtonian or Lagrangian approach, depending on the output quantities of interest.

For the ICA, an arbitrary joint in the closed loop is virtually separated, using a virtual generalized displacement, into two components fixed in each of the two connecting bodies. The virtual displacement (in a direction not normally allowed by the ideal joint), when multiplied by an arbitrarily large constant, approaches in value the internal generalized reaction force of the ideal joint. This internal joint reaction force, expressed in terms of the virtual relative displacement within the joint, is treated in subsequent analysis as an external force and/or moment acting on the system. (In fact there is a pair of equal and opposite reaction forces/moments acting on the two bodies connected by the virtually displaced joint.) At this point the system can be analyzed using any approach the analyst prefers for the problem at hand. The introduction of the joint reaction force/moment yields a set of differential equations that implicitly satisfy the kinematic closure constraint equations. The overall system descriptive dynamic equations

thus remain in their natural unconstrained form, whether they are derived using a Newtonian, Lagrangian or any other approach.

Since the ICA utilizes virtual displacements that violate the rigid joint constraint assumption, it invites comparison to the *method of virtual work* commonly used in solving statics problems [7] and sometimes dynamics problems [8]. While both approaches share the assumption of a virtual displacement, which might correspond to an actual displacement, the ICA does not directly yield the system dynamics equations as is the case for the method of virtual work. The ICA provides an alternative to forming *explicit* kinematic system constraint equations. The ICA could be used in conjunction with the method of virtual work, but it also lends itself to use with both the Lagrange and Newtonian approaches to developing the overall system dynamics equations.

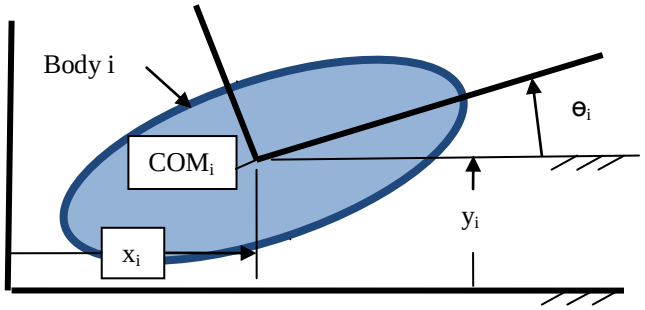

Figure 2 Coordinates Describing the Position of a Planar Body

4 REVOLUTE JOINT AXIOM

To illustrate the ICA, a simple situation is considered. Figure 2 shows a body, i, confined to planar motion. The body has a coordinate system fixed in the body and located at the body's center of mass, COM. The location of the body is determined uniquely by the location, (x_i,y_i), of its COM and the rotation, θ_i, of a body fixed axis with respect to a globally fixed axis. A pin-type (revolute) joint connects the body to an adjacent body in the kinematic chain. (The method will work with more and different types of joints, but a simple model of two planar bodies connected by a pin joint is sufficient for an introduction to the ICA.) Figure 3 shows bodies g and h that make up part of a kinematic chain. The (pin) joint connecting the bodies is shown separated into its two parts, each part being fixed in its respective connected body. The ***First Axiom*** in the ICA is that the pin joint reaction force between connected bodies can be represented by a constant, k, times the virtual linear displacement between the joint centers in the two connected bodies. Thus the joint contact force, Fk, resulting from body h acting on body g can be written as

$$Fk_{hgx} = k\,(p_{bx} - p_{ax}) \quad (1)$$
$$Fk_{hgy} = k\,(p_{by} - p_{ay}) \quad (2)$$

where

$$p_{ax} = x_g + r_a \cos(\theta_g + \xi_a) \quad (3)$$
$$p_{ay} = y_g + r_a \sin(\theta_g + \xi_a) \quad (4)$$
$$p_{bx} = x_h + r_b \cos(\theta_h + \xi_b) \quad (5)$$
$$p_{by} = y_h + r_b \sin(\theta_h + \xi_b) \quad (6)$$

and the subscripts a and b refer to the points at the pin centers in bodies g and h respectively.

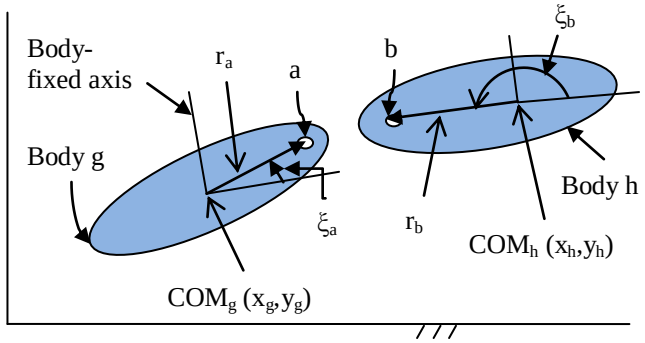

Figure 3 Virtually Separated Revolute Joint

While the stiffness of any particular bearing is nonlinear over a wide range of radial loads and angular speeds [12], for any given loading situation the value of k is usually reasonably constant. Rolling element bearings for revolute joints typically have a radial stiffness value, k, in the range of $1*10^8$ to $30*10^8$ N/m. Lubricated journal bearings have k's in the range $1*10^7$ to $10*10^7$ N/m [12,13].

In addition, if one uses numerical integration programs that do not intrinsically have a numerical damping effect on higher frequency motion, it is important to explicitly add a joint damping force (where the symbol ′ means the derivative with respect to time)

$$Fd_{hgx} = c\,(p'_{bx} - p'_{ax}) \quad (7)$$
$$Fd_{hgy} = c\,(p'_{by} - p'_{ay}) \quad (8)$$

where c is an arbitrary constant (of magnitude much less than k) and

$$p'_{ax} = x'_g - \theta'_g\, r_a \sin(\theta_g + \xi_a) \quad (9)$$
$$p'_{ay} = y'_g + \theta'_g\, r_a \cos(\theta_g + \xi_a) \quad (10)$$
$$p'_{bx} = x'_h - \theta'_h\, r_b \sin(\theta_h + \xi_b) \quad (11)$$
$$p'_{by} = y'_h + \theta'_h\, r_b \cos(\theta_h + \xi_b) \quad (12)$$

Real joints have a certain amount of intrinsic damping. In fact it is much greater than that needed to remove parasitic high frequency vibration effects in ICA modeling. Thus including a small damping term in the joint model not only produces a better behaved simulation model, it is also in the spirit of performing an analysis that approaches real behavior as opposed to ideal behavior. The damping coefficient, c, for radial motion in a typical rolling element bearing is in the range $2*10^3$ to $10*10^3$ N-s/m [12,13]. For lubricated plain journal bearings the radial damping coefficient is in the range $1*10^5$ to $5*10^5$ N-s/m.

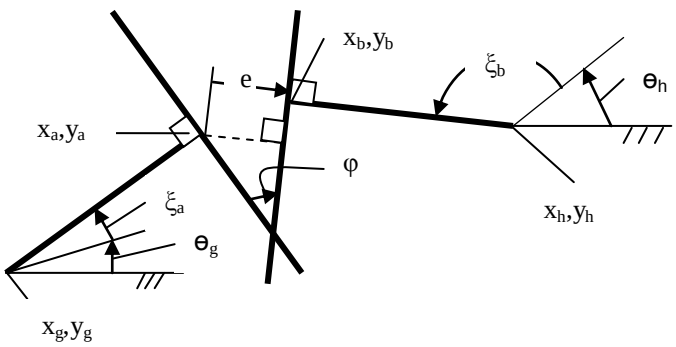

Figure 4 Virtually Separated Prismatic Joint

5 PRISMATIC JOINT AXIOM

Figure 4 shows two planar bodies, g and h, connected by a planar prismatic (sliding) joint. The prismatic joint can be imagined to be virtually flexible so that the sliding axes in the two bodies are laterally separated and rotated with respect to each other as shown. The connecting members are defined relative to their centers of mass (at x_g,y_g and x_h,y_h). The location of the prismatic joint within the body is given by its perpendicular distance from the mass center. The point on the sliding axis where the perpendicular from the body mass center intersects it is the characteristic point for the prismatic joint in that body. This is analogous to the center of the revolute joint. The ***Second Axiom*** of the ICA is that planar prismatic joint forces can be represented by a force and moment acting at the characteristic point of one of the bodies. The force is equal to a constant k1 multiplied by the perpendicular distance, e, from the characteristic point to the sliding axis in the connecting body joint. The moment is equal to a constant k2 multiplied by the angular displacement, φ, between the two sliding axes. For a virtual analysis, or one for which the real ki values are not known, the two ki's are arbitrarily large and can be numerically the same or different.

To evaluate the expressions for the separation distance described in the Second Axiom, let the global coordinates of the first body, g, characteristic point be x_a and y_a and the global coordinates of the connected body, h, characteristic point be x_b and y_b. Each characteristic point can be expressed in terms of the body center of mass coordinates in a manner similar to equations (3) – (6). Using basic planar geometry, the separation distance, e, described in the Second Axiom is

$$e = (x_a - x_b)\, cb + (y_a - y_b)\, sb \tag{13}$$

with $sb = \sin(\theta_h + \xi_b)$ and $cb = \cos(\theta_h + \xi_b)$. This in turn is resolved into the x and y directions to get joint forces

$$Fk_{hgx} = - k1\, e\, cb \tag{14}$$
$$Fk_{hgy} = - k1\, e\, sb \tag{15}$$

The angular displacement, φ, described in the axiom, yields a CCW joint reaction moment on body g

$$Mk_{hg} = k2 \bullet \varphi \tag{16}$$

where $\varphi = \theta_h + \xi_b - \theta_g - \xi_a - \pi$ (17)

and π has its usual numeric meaning in radians.

As with the revolute joint, to avoid parasitic vibratory motions, in addition to the Second Axiom forces, joint damping forces must be introduced in the form

$$Fd_{hgx} = -c1\, e'\, cb \tag{18}$$
$$Fd_{hgy} = -c1\, e'\, sb \tag{19}$$
$$Md_{hg} = c2\, (\theta_h' - \theta_g') \tag{20}$$

where e′ is the time derivative of e, given by expression (13), and the ci's are constants which are small relative to the ki's. The magnitude of the stiffness and damping coefficients for linear bearings is of a similar magnitude to that for rotary bearings [12,14]. However the particular geometry of the linear bearing can have a significant effect on the magnitude of the rotary stiffness and damping coefficients.

6 FORMING THE EQUATIONS OF MOTION

The steps to forming the equations of motion for a particular mechanism depend on the analyst's intent. If the analyst is only interested in the mechanism's kinematic performance, with no interest in joint reaction forces, then only one mechanism joint needs to be considered virtually separated, and the classic Lagrangian approach can be used for the mechanism analysis using the analyst's preferred choice of kinematic coordinates (absolute, relative or a combination). If a few of the joint reaction forces are of interest, then those joints must be considered virtually separated and the Lagrangian approach used. However if most or all of the joint forces are of interest, then all joints should be considered virtually separated and the classic Newtonian approach is preferred (but not required). In summary, to use the ICA the Lagrangian approach can always be used with at least one of the joints virtually separated; whereas the Newtonian approach can only be used if all of the joints are virtually separated.

For linkages, the need for explicit loop closure constraint equations is eliminated if at least one joint in the loop is represented by a virtually flexible joint as described above. The approach is also useful for analyzing mechanisms where one or more joints are designed to be flexible and the joint virtual displacements are in fact real displacements controlled by known joint flexibility and damping characteristics (represented by the constants k and c).

One other approach in the literature, the numerical analysis based penalty function approach [15], yields a system of equations similar in resulting form to that of the ICA in that the constraint conditions are accounted for in the dynamic equations and not in explicit additional equations. (All other current approaches involve the formation of separate closure constraint equations of some form.) More recently Cuadrado et al [6] combined the penalty function approach with a recursive

formulation that claims superior performance to other methods under certain circumstances.

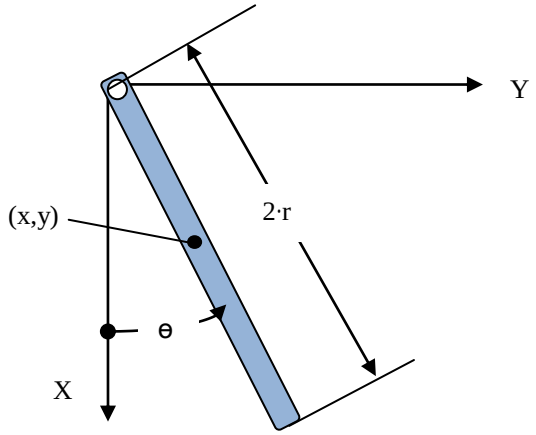

Figure 5 Large Amplitude Simple Pendulum

7 LARGE AMPLITUDE PENDULUM

Consider the simple large amplitude pendulum shown in Figure 5, with an ideal pin joint connecting a uniform rod, of mass m and length 2·r, to an inertial reference frame. The system has one degree of freedom and by inspection the system is usually (and most simply) described using the single coordinate ɵ. The single system equation of motion using the angle ɵ as the generalized coordinate takes the form (where the symbol ″ refers to the second derivative with respect to time)

$$J\,\theta'' + m\,g\,r\,\sin(\theta) = 0 \qquad (21)$$

where $J = (m/3)\,(2\,r)^2$. Starting with initial conditions for ɵ and ɵ′, (21) can be numerically solved for ɵ and ɵ′ as time t increases. At any given time the ideal joint reaction force acting on the pendulum is

$$F_x = -m\,\{\,g + r\,[\theta'^2\cos(\theta) + \theta''\sin(\theta)]\} \qquad (22)$$
$$F_y = m\,r\,[\theta''\cos(\theta) - \theta'^2\sin(\theta)] \qquad (23)$$

Applying the ICA to this problem yields the dynamic equations (with u_1=x, u_3=y, and u_5=ɵ)

$$u_1' = u_2 \qquad (24)$$
$$u_2' = (1/m)\,[mg + k\,(p1_x - p2_x) + c\,(p1'_x - p2'_x)] \qquad (25)$$
$$u_3' = u_4 \qquad (26)$$
$$u_4' = (1/m)\,[k\,(p1_y - p2_y) + c\,(p1'_y - p2'_y)] \qquad (27)$$
$$u_5' = u_6 \qquad (28)$$
$$u_6' = (4*r/J)\,\{k\,[(p1_x - p2_x)\sin(u_5) - (p1_y - p2_y)\cos(u_5)] + c\,[p2'_y\cos(u_5) - p2'_x\sin(u_5)]\} \qquad (29)$$

where $p1_x$ = constant, $p1_y$ = constant, $p1'_x = p1'_y = 0$,

$$p2_x = u_1 - r\cos(u_5) \qquad (30)$$
$$p2_y = u_3 - r\sin(u_5) \qquad (31)$$
$$p2'_x = u_2 + r\,u_6\sin(u_5) \qquad (32)$$
$$p2'_y = u_4 - r\,u_6\cos(u_5) \qquad (33)$$

For a sample problem where the origin is placed at the pendulum pivot, m = 1 kg, r = 0.5 m, and the initial conditions that $\theta = \pi/2$ and $\theta' = 0$, the problem has been solved using the basic Runge Kutta numerical integration function (ode45) in the commercial program MATLAB (version 7.5). The same commercial integration program was used for solving equation (21) (converted to two first order ODEs) and the ICA equations (24)-(29). (Representative rolling element bearing parameters of $k = 2*10^8$ N/m and c = 3500 N-s/m were used in the ICA calculations.) Figure 6 shows the plots for F_x, F_y and ɵ for the first 2 seconds using both methods, the results differing by less than $5*10^{-5}$ units for both the angle and joint reaction forces (Figure 7). This means that the two solutions agree to within six significant figures even though the relative error tolerance specified in the numerical integration routine is only $1*10^{-5}$. For practical purposes the results are identical.

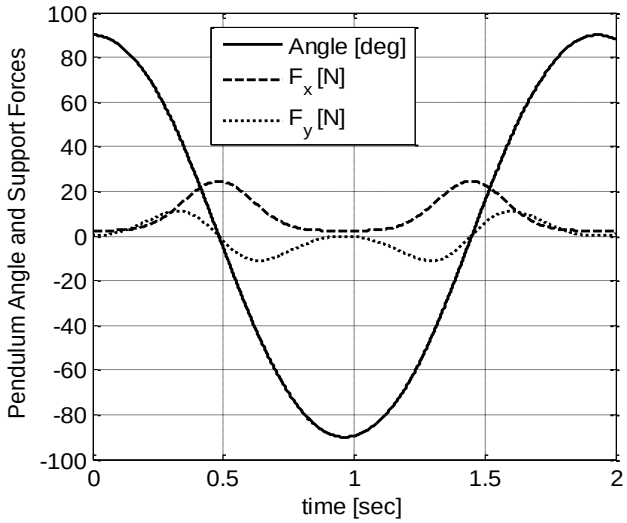

Figure 6 Results for Sample Pendulum Problem

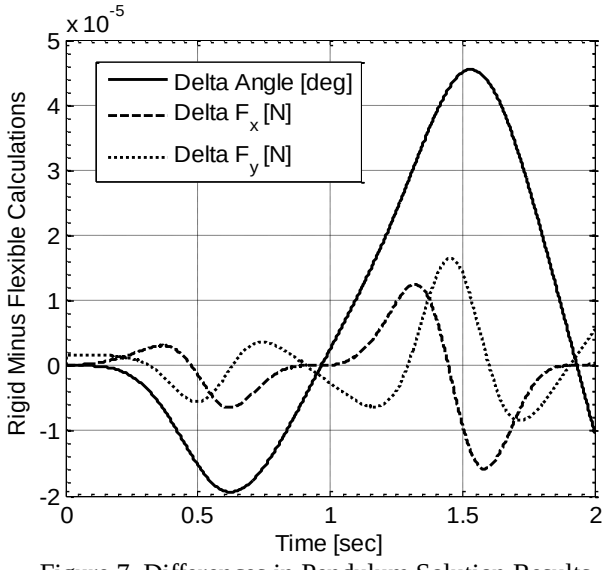

Figure 7 Differences in Pendulum Solution Results

It is interesting that the $p2'_x$ and $p2'_y$ terms are a very small part of the final solution (using the ICA) with Fd/Fk< $1*10^{-4}$ throughout the range of motion when c=3500 N-m/s. Essentially the same results (with visually identical plots like Figure 6 and 7) can be produced with very low c values, which result in even smaller Fd/Fk ratios. Even with a c value of 0, MATLAB ode45 (which is known to have some inherent numerical damping) is stable and the parasitic force vibrations have a peak to peak amplitude of less than $3*10^{-6}$ N, so the kinematic and joint force plots look identical for c=0 and c=3500 N-s/m. While all the popular MATLAB equation solvers get similar results when realistic c values are used, when c is set to zero their performance differs. As noted above, when ode45 is used very small high frequency vibrations are superposed on the previous results. When ode23t (which is supposed to be good for stiff problems and to not have numerical damping) is used, there are no observed high frequency vibrations, but the results are different at the 5^{th} significant figure. (The lack of vibrations with no numerical damping is a surprise.) When ode15s is used (which is supposed to be good for stiff problems) there are no high frequency vibrations, but the results are only accurate to one significant figure. Using ode113, a variable order Adams-Bashforth-Moulton PECE solver for nonstiff systems, yields accurate results with parasitic high frequency vibrations of about 1*10-5 in maximum peak to peak amplitude. It appears that the nonstiff MATLAB differential equation solvers work better on this problem than the stiff equation solvers, with the usual first choice, ode45, an explicit Runge-Kutta (4,5) method, working consistently the best.

The point of this example is not that the ICA represents a better approach for solving this pendulum problem, but rather that it achieves results similar to those achieved by assuming ideal joint behavior and that the ICA equations are well-behaved.

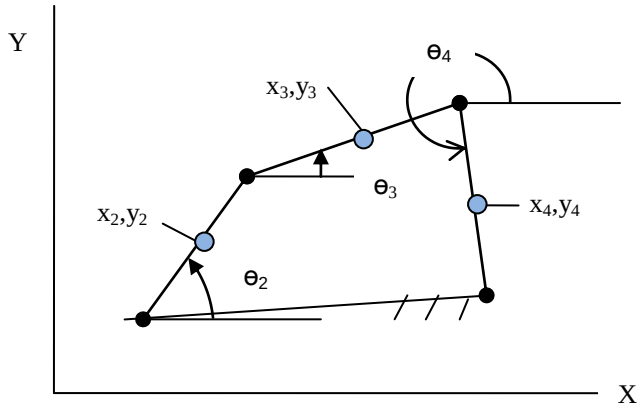

Figure 8 Planar Four-Bar Coordinates

8 PLANAR FOUR-BAR

A planar four-bar is shown in Figure 8 with the link variables defined as illustrated. (Gravity is not considered in the problem.) If the joint reaction forces are not of interest, then a Lagrangian approach can be taken for both the ICA and traditional methods. The traditional method would require the formation of two constraint equations, typically be converted to a nonholonomic form by differentiation, to represent mechanism closure, while the kinetic energy of the system is expressed in terms of link angles θ_i (which could be relative or absolute). The usual approach uses Lagrange multipliers to combine the constraint equations [9] with the energy derived dynamic equations. Chace [4] was probably the first to apply this (now common) general approach to mechanisms dynamic analysis problems. The resulting augmented dynamic equations and the constraint equations can be solved for the link positions and velocities using a numerical integration computer routine. If the joint forces are needed, then one would typically first solve the kinematic problem using an approach like the Lagrange approach and then back solve for the joint forces using Newtonian equilibrium equations for each link [16]. Many other approaches have been suggested and used for calculating joint reaction forces with [17] being an interesting recent one.

For the planar four-bar and the ICA method, if joint reactions are not of interest, then one would write a set of Lagrangian equations of the first kind [7,8,9] with the only nonconservative generalized active forces coming from the small (in relative magnitude) joint damping terms. This leads to a set of 3 second order differential equations in the variables θ_2, θ_3, and θ_4, which by the usual substitutions [8] can be converted to a system of six first order differential equations in terms of the link angles and their angular velocities. There is no need for constraint equations or Lagrangian multipliers and were it not for the small joint damping term, the equations could be derived solely from the Lagrangian energy expression. Thus, while the ICA in the case of the pendulum problem generates a larger system of equations, in the case of the planar four-bar, the system to be solved takes on a simpler form because there is no need to specifically accommodate nonholonomic constraint equations or to add the Lagrange multiplier variables, both of which often can be a source of complication for the numerical integration routines. The Lagrange multipliers in mechanisms analysis problems are usually generalized active forces (and indeed Kane [8] treats them directly as such without introducing the multiplier concept), so the ICA has the possibly valuable feature of having all its generalized coordinates being displacements rather than being a mixture of displacements and forces.

The ICA combined with the Lagrangian approach certainly is intriguing for certain problems and should continue to be explored. However the author is also interested in examining the special case where all the joints are considered to be flexible (not ideal) and Newtonian equations of equilibrium are applied to each mechanism member without using generalized

coordinates that cleverly reduce the number of coordinates needing to be evaluated. With the ICA and the planar Newtonian approach, each additional body in the system adds three equilibrium equations and three global coordinates, which makes subsequent analysis quite straightforward. The equilibrium equations for each body involve only the coordinates associated the body and its directly connected bodies. Thus systems with multiple loops (or even open loops) and multiple degrees of freedom can be handled in a straight-forward manner, where each additional body brings with it both three new kinematic coordinates and three new independent dynamic equations.

For the planar four-bar this Newtonian approach leads to a set of nine second order differential equations in terms of the nine unknowns (x_i,y_i,θ_i), i=2,3,4, which can be converted to 18 first order differential equations using the usual substitutions [8]. The equilibrium equations for each link have the same pattern and once the system is solved for the kinematic variables, the joint reaction forces can be directly calculated without any further equation solving. Such a pattern is ideal for adapting the method for use in a general purpose computer program where many mechanism topologies can be handled. Also the approach can easily handle systems with multiple degrees of kinematic freedom where the nature of external forces, controlling forces and member inertias (along with the mechanism topology) determine the mechanism kinematics.

The system of 18 equations for the four-bar using the ICA has been programmed into a MATLAB function. Some typical problems have been tried and the results agree with those found using conventional methods on four-bar models with ideal joints. One problem of particular interest (that can be easily used for comparison with other methods) is one where a four-bar is driven by a motor with very high apparent inertia so that the input crank angular velocity is close to constant. The inputs are (with consistent units assumed) m2=1, m3=4, m4=4, I2=100000, I3=5.333, I4=5.333, r2=0.5, r3=0.5, r4=2, r5=2, r6=2, r7=2, ξ2=π, ξ3=0, ξ4=π, ξ5=0, ξ6=π, ξ7=0, the follower ground pin location is x=6, y=0 relative to the crank ground pin, and the crank angular velocity is initially 10 rad/sec CCW. For the ICA the stiffness constant was selected to be $1 * 10^8$ and the damping constant to be 3000. The results have been compared with the commercial program, Working Model 2D, and the student edition of Norton's Fourbar program (provided with reference [18]). Figure 9 shows the output ground link reaction force components, $F14_x$ and $F14_y$, as a function of the crank angle θ_2. All three program results agree within the noise level generated by numerical integration.

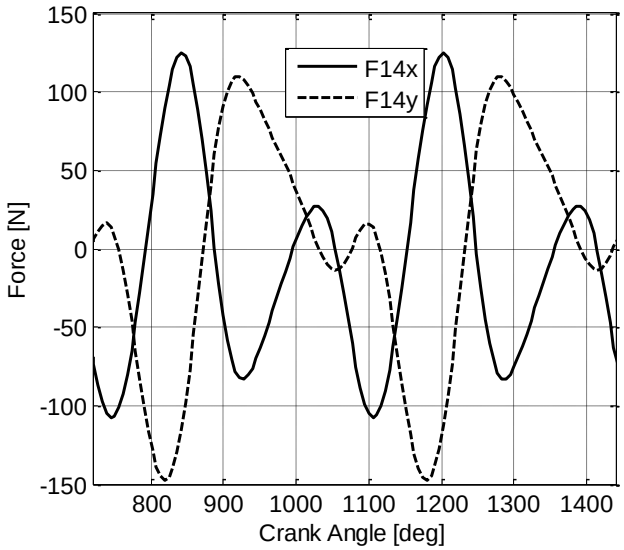

Figure 9 Example Planar Fourbar Follower Ground Pin Reaction Forces vs. θ_2 [deg]

A variety of planar fourbar problems have been tried using the ICA. For examples using values of k and c that are similar to commercial bearings (that are not intended to flexible) the joint reaction force and kinematic results are the same (to within 4 significant figures for the forces and even closer for displacements and velocities) as those using ideal infinitely stiff bearings. However if bearings with some intended flexibility are used, like journal bearings that are rubber mounted in a housing, the results vary quite substantially, particularly the joint reaction forces. For cases where k and c are not known, the procedure is to set c to a nominal very light value, like 500 N-s/m, and then successively increase k (starting at $1*10^6$ N/m) by a factor of 10 until the desired results agree to within the significant figures desired. Using MATLAB numerical integration routines, as the k values exceed $1*10^6$ N/m, it is important to specifically limit the integration step-size to less than one tenth of the specified time step needed for plotting smooth results. If only the kinematic properties are of interest, the starting k value can be lowered to $1*10^5$ N/m. Since the analyst typically does not know a consistent set of initial conditions (ICs) for more complex problems, the solution method needs to be able to easily handle situations with inconsistent initial conditions, like the coupler and follower initial velocities being zero and the crank initial velocities being the desired quantities. If c is initially set to zero, substantially inconsistent ICs will induce system vibrations that are significant relative to the solution that would occur if consistent ICs were chosen. However if c values are chosen that correspond to even the most lightly damped real bearings, the IC-induced initial vibrations are quickly damped out and the desired solution is all that appears after a few degrees of crank rotation.

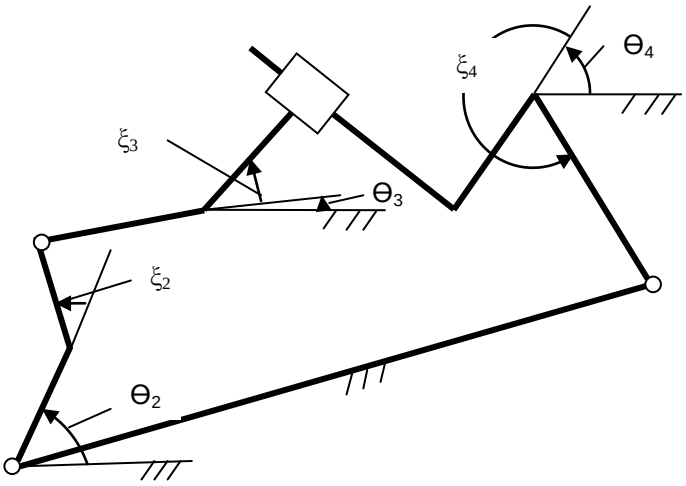

Figure 10 Generalized Inverted Slider Crank Mechanism

9 INVERTED SLIDER CRANK

A MATLAB function has been written that evaluates the 18 first order ODEs resulting from applying the ICA to solve inverted slider crank problems of the form illustrated in Figure 10. In describing the mechanism topology, the convention is taken that the body-fixed x-axis aligns with the line from the joint characteristic point to the body mass center, starting first with the left ground revolute joint and then working clockwise around the mechanism, as is shown in Figure 10. Sample numerical problems have been run using the ICA-derived MATLAB inverted slider crank function and then compared with the results using Working Model 2D (with ideal joints). Again the results are identical for both the ICA program and the commercial program, which also uses a variant of Runge Kutta integration, to within the usual tolerances for numerical integration.

10 OBSERVATIONS

Besides using the stiffness, k, and damping, c, values that corresponded to typical real bearings, a number of simulations have been run using a variety of k and c values. Changing c values within a reasonable range has no effect on the results. However if very small values are used (like less than .001 times a typical commercial bearing damping coefficient value, which in turn is just a few percent [12] of the critical damping coefficient for a typical application) parasitic vibrations start to appear that would not occur in a real system.

Changing k values did have an effect on the results. For example, the pendulum problem, using the nominal k of $2*10^8$ N/m, required setting the MATLAB ODE solver maximum step size to $1*10^{-5}$. Otherwise the routine generated results that were obviously in error. The fourbar and inverted slider crank problems were not as sensitive as the pendulum and the default ode45 step-size calculation worked fine for k values similar to those for real bearings. For k's in the range $1*10^5$ to $1*10^6$ N/m, correct results (that were similar to the higher k results) were obtained using the default MATLAB variable step size setting. However, as k was reduced to less than $1*10^5$ N/m, the joint flexibility effect started to be noticeable in the results. Of course a real system with such flexible joints would probably exhibit similar behavior, but a flexible bearing mechanism test apparatus has not been built yet (by the author) to test that assumption.

All the standard (both regular and stiff) numerical integration routines supplied within MATLAB worked well using a one-year-old Dell Latitude D820 laptop computer to solve the problems. (The author also tried the standard MathCAD ODE integration routines and they also worked well.) However in future work with complex spatial mechanisms, it is expected that substantial gains can be made by using special numerical integration algorithms [19,20] that accommodate the large range of system natural frequencies (that result from allowing bearing flexibility) and also take advantage of the sparse banded nature of the ICA generated system ODEs.

11 CONCLUSIONS

The ICA provides a viable alternative to using explicit kinematic constraint equations as auxiliary equations to the basic dynamic equilibrium equations for modeling multi-body mechanical systems. The pendulum and linkage examples illustrate the use of the ICA for some common modeling problems. The introduction of the small joint damping terms enables the integration to proceed smoothly for the problems tested. The magnitude of the damping coefficient necessary to remove parasitic vibrations is of much lower magnitude than that for lightly damped real bearings. Also the damping terms turn out to be an insignificant component of the kinematic and force results. They only play a significant role at the analysis startup if the initial values for the generalized coordinates are inconsistent and/or not physically possible. After the system stabilizes from its "rough" start, the damping terms become insignificant in magnitude.

The ICA can be used in mechanism topology dependent ways with the Lagrangian approach to produce a relatively compact model, if the joint reaction forces are not of immediate interest. However for the general planar modeling case, where any combination of kinematic and force quantities may be of interest, the ICA in combination with the basic body-by-body Newtonian approach, has three distinct advantages over the current practice of using kinematic constraint equations. Firstly it naturally generates a system of differential equations with a diagonal mass coefficient matrix. Secondly the governing differential equations are not further complicated by an auxiliary set of algebraic equations or their derivatives. Thirdly nonideal joint characteristics are naturally included in the model.

NOMENCLATURE

COM_a	center of mass of body *a*
Fd_{hg}	reaction force between bodies *h* and *g* due to joint damping
Fk_{hg}	reaction force between bodies *h* and *g* due to joint stiffness
Md_{hg}	reaction moment between bodies *h* and *g* due to joint damping
Mk_{hg}	reaction moment between bodies *h* and *g* due to joint stiffness
c	joint damping coefficient
k	joint stiffness coefficient
p_{ax}	x-coordinate of the position of joint characteristic point *a* w.r.t. an inertial reference frame
q'	time derivative of parameter *q*
r_i	distance from the body COM to characteristic point *i* (which is fixed in the same body and associated with a particular joint)
u_i	generalized state-space coordinate *i*
x_i	x-coordinate of the COM of body *i* w.r.t. an inertial reference frame
y_i	y-coordinate of the COM of body *i* w.r.t. an inertial reference frame
θ_i	angle of the body-*i*-fixed x-axis w.r.t. the x-axis of the inertial reference frame
ξ_i	angle of joint position vector r_i w.r.t. the body-fixed x-axis

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