Explicit PL self-knottings and the structure of PL homotopy complex projective spaces

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THE STRUCTURE OF PL HOMOTOPY COMPLEX
PROJECTIVE SPACES

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We show that certain piecewise-linear homotopy complex projective spaces may be described as a union of smooth manifolds glued along their common boundaries. These boundaries are sphere bundles and the glueing homeomorphisms are piecewise-linear self-knottings on these bundles. Furthermore, we describe these self-knottings very explicitly and obtain information on the groups of concordance classes of such maps.

A piecewise linear homotopy complex projective space $\widetilde{CP}^n$ is a compact PL manifold $M^{2n}$ homotopy equivalent to $CP^n$. In [22] Sullivan gave a complete enumeration of the set of PL isomorphism classes of these manifolds as a consequence of his Characteristic Variety theorem and his analysis of the homotopy type of $G/PL$. In [15] Madsen and Milgram have shown that these manifolds, the index 8 Milnor manifolds, and the differentiable generators of the oriented smooth bordism ring provide a complete generating set for the torsion-free part of the oriented PL bordism ring. Hence a study of the geometric structure of these exotic projective spaces $\widetilde{CP}^n$, especially with regard to their smooth singularities, may extend our understanding of the PL bordism ring. This paper begins such a study in which we obtain a geometric decomposition of $\widetilde{CP}^n$, into (for many cases) a union of smooth manifolds glued together by PL self-knottings on certain sphere bundles. We also obtain information on groups of concordance classes of PL self-knottings from these decompositions and a number of fairly explicitly constructed examples of self-knottings. I would like to thank by thesis advisor R. J. Milgram for many helpful discussions.

I. Sullivan's classification of PL homotopy $\widetilde{CP}^n$ proceeds as follows: Given a homotopy equivalence $h: \widetilde{CP}^n \rightarrow CP^n$ make $h$ transverse regular to $CP^j \subset \widetilde{CP}^n$, the standard inclusion. The restriction of $h$ to the transverse inverse image $h^{-1}(CP^j) = N^{2j} \subset \widetilde{CP}^n$ is a degree one normal map
with simply connected surgery obstruction
\[ \sigma_j \in P_{2j} = \begin{cases} Z, & j \text{ even} \\ Z/2Z, & j \text{ odd} \end{cases} \]

For \( j = 2, \ldots, n - 1 \) these obstruction invariants yield a complete enumeration—i.e. the set of PL isomorphism classes of \( \widetilde{CP}^n \) is set-isomorphic to the product \( Z \times Z_2 \times Z \times \cdots \times P_{2(n-1)} \) with \( n - 2 \) factors.

We will use the following notation to specify elements with this classification:

\[ \widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \]

will denote the PL homotopy \( \widetilde{CP}^n \) with invariants \( \sigma_j \in P_{2j} \) in Sullivan's enumeration.

We recall that a PL homeomorphism \( f: M \to M \) is a “self-knotting” and \( M \) is said to be “self knotted” if \( f \) is homotopic but not PL isotopic to the identity. Also, PL homeomorphisms \( f, g: M \to M \) are “PL concordant” (pseudo-isotopic) if we have a PL homeomorphism \( F: M \times I \to M \times I \) with \( F(m, 0) = (f(m), 0) \) and \( F(m, 1) = (g(m), 1) \) for \( m \in M \). We then define:

\[ (2) \ SK(M) = \text{“the group (under composition of maps) of PL concordance classes of PL self-knottings of } M.\]"

Unless otherwise noted \( CP^j \subset CP^n \) means the standard embedding of \( CP^j \) onto the first \( (j + 1) \) homogeneous coordinates of \( CP^n \) or a smooth ambient isotope of this embedding. In this context we define:

\[ (3) \ \nu_N(CP^j) = \text{“the smooth tubular disc bundle neighborhood of the embedding } CP^j \subset CP^n.\]

Our results are as follows:

**Theorem A.** For \( n \geq 4 \) and \( \sigma_2 \equiv 0 \) (2) every \( \widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \)

is PL homeomorphic to the identification space

\[ [\widetilde{CP}^n - \nu_n(CP^1)] \cup_{\varphi_{n-1}} [\nu_n(CP^1)] \]

where \( \widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-2}, 0) \) and the identification is over a PL homeomorphism

\[ \varphi_{n-1}: \partial \nu_n(CP^1) \to \partial \nu_n(CP^1). \]

We prove Theorem A in Part II by a careful description of Sullivan's classification and an easy \( h \)-cobordism argument.
An immediate consequence of Theorem A is the decomposition of $\widetilde{CP}^{n+1}$ into 
$\widetilde{CP}^{n+1} = [CP^{n+1} - \nu(CP^1)] \cup_{\varphi_0}[\nu(CP^1)]$.

**Theorem B.** For every $n \geq 4$ and non-zero $\tau \in P_{2n}$ there is a PL self-knotting 
$\varphi_\tau: \partial \nu_{n+1}(CP^1) \to \partial \nu_{n+1}(CP^1)$

which will suffice for the glueing homeomorphism in Theorem A.

We establish this theorem by an explicit construction of $\varphi_\tau$ in Part III.

**II.** Here we prove Theorem A by beginning with a construction which shows how to obtain $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ from $\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1})$ for $n \geq 4$:

Let $h: \widetilde{CP}^n \to CP^n$ be a homotopy equivalence, and let $M^{2n}$ be the compact $(n - 1)$-connected Milnor or Kervaire manifold of Index $8\sigma_n$ or Kervaire-Arf invariant $\sigma_n$ as the case may be [4]. Let $r: M^{2n} \to S^{2n}$ be a degree one map. Then $h\#r: \widetilde{CP}^n\#M^{2n} \to CP^n\#S^{2n} = CP^n$ is a degree one normal map with 1-connected surgery obstruction $\sigma_n$. We define $\hat{H}$ as the $D^2$ bundle over $\widetilde{CP}^n\#M^{2n}$ induced by $h\#r$ from $H$, the disc bundle associated to the complex line bundle over $CP^n$. Let $\hat{h}: \hat{H} \to H$ be the bundle mapping. We note that the map $h\#r$ is $(n - 1)$-connected with homological kernel $K_n = \pi_n(M_0^{2n})$ where $M_0^{2n} = M^{2n} - D^{2n}$. The bundle $\hat{H}$ is trivial over $M_0^{2n}$ since $M_0^{2n} = (h\#r)^{-1}(point)$. In $M_0^{2n} \times D^2$ we can represent $\pi_n(M_0^{2n})$ by disjointly embedded spheres $S^n \subset M_0^{2n} \times S^1$ with trivial normal bundles. This follows by general position and the fact that the normal bundles of the generating spheres $S^n \subset M_0^{2n}$ are the stably trivial tangent disc bundles $\tau(S^n)$. We now attach a solid handle $D^{n+1} \times D^{n+1}$ along $S^n \times D^{n+1} \subset M_0^{2n} \times S^1$ for each generator of $\pi_n(M_0^{2n})$ and extend the map $\hat{h}$ across these bundles. This is possible since the embedded spheres are in the homotopy kernel of $\hat{h}$. Call the resulting PL manifold $\tilde{H}$ and the extended map $\tilde{h}: \tilde{H} \to H$. In the process of extending $\hat{h}$ across the handles, we may guarantee that $\tilde{h}$ is a map of pairs $(\tilde{H}, \partial) \to (H, \partial)$. We observe, then, the:

**Proposition.** $\tilde{h}: (\tilde{H}, \partial) \to (H, \partial)$ is a homotopy equivalence of pairs.
This follows directly from the construction as \( \tilde{H} \) deformation retracts onto \( \tilde{C}P^n \# M^{2n} \cup \{ e^n \} \) where the \( n \)-cells \( e^n \) are attached so as to kill the entire homology kernel of \( (h \# r) \). Hence \( \tilde{h}: \tilde{H} \to H \) is a homology isomorphism, and as \( \tilde{H} \) is 1-connected we have by Whitehead's theorem that it is a homotopy equivalence. The restriction of \( \tilde{h} \) to the boundary is likewise a homology isomorphism as the boundaries, \( D^{n+1} \times S^2 \), of the solid handles are precisely the surgeries needed to cobord \( \tilde{h}: \partial \tilde{H} \to \partial H \) to a homotopy equivalence.

In particular as \( n \geq 3 \) we note that the boundary manifold, \( \partial \tilde{H} \), is a PL \((2n+1)\)-sphere by the Poincaré conjecture. Thus, we attach \( D^{2n+2} \) to \( \tilde{H} \) as the PL cone on \( \partial \tilde{H} \) and define:

\[
\tilde{C}P^{n+1} = \tilde{H} \cup c(\partial \tilde{H}) \quad \text{and} \quad h: \tilde{C}P^{n+1} \to C P^{n+1} = H \cup c(\partial H)
\]

by radial extension of \( \tilde{h} \) into \( c(\partial \tilde{H}) \).

Observe that \( h \) has "built-in" transverse inverse image \( \tilde{C}P^n \# M^{2n} = h^{-1}(C P^n) \) with surgery obstruction \( \sigma_n \). Hence, this \( \tilde{C}P^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \) is the space we require.

Now, given \( \tilde{C}P^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}) \) let us consider a bit more closely the suspension and generalized suspension constructions described above. First, assume the homotopy equivalence

\[
h: \tilde{C}P^n \to C P^n
\]

is the identity map on a disc \( D^{2n} \subset \tilde{C}P^n \). Let \( \tilde{C}P^n = \tilde{C}P^n - D^{2n} \), \( M^{2n}_0 = M^{2n} - D^{2n} \) and observe that \( \tilde{C}P^n \# M^{2n} = \tilde{C}P^n_0 \cup_\partial M^{2n}_0 \). Now, let \( \tilde{C}P^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0) \) be the suspension\(^1 \) of \( \tilde{C}P^n \) with homotopy equivalence

\[
\tilde{h}: \tilde{C}P^{n+1} \to C P^{n+1}
\]

and \( \tilde{C}P^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \) be the general suspension of \( \tilde{C}P^n \) with homotopy equivalence

\[
\tilde{h}: \tilde{C}P^{n+1} \to C P^{n+1}.
\]

Let \( D^{2n} \subset C P^n \) be the image \( h(D^{2n}) \) and let \( C P^1 = S^2 \subset C P^{n+1} \) be represented as \( D^2_\ast \cup c(\partial D^2_\ast) \) in \( C P^{n+1} = H \cup c(\partial H) \) with \( D^2_\ast \) the fiber in \( H \) over the center of the disc \( D^{2n} \). Then \( \nu_{n+1}(C P^1) \subset C P^{n+1} \) may be represented as the set \( D^2_\ast \times D^{2n} \cup c(\partial H) \), a \( D^{2n} \) bundle over the sphere \( S^2 = D^2_\ast \cup c(\partial D^2_\ast) \).

Now let \( \tilde{V} = \tilde{h}^{-1}(\nu_{n+1}(C P^1)) \) and \( \hat{V} = \hat{h}^{-1}(\nu_{n+1}(C P^1)) \) in \( \tilde{C}P^{n+1} \) and \( \hat{C}P^{n+1} \), respectively. We observe directly from the constructions that

\(^1\)We say \( \tilde{C}P^{n+1} \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, 0) \) in the "suspension" of \( \tilde{C}P^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \) as it is precisely the Thom complex of the line bundle induced over \( \tilde{C}P^n \).
\( \hat{C}P^{n+1} - \hat{V} \) and \( \hat{C}P^{n+1} - \hat{V} \) are precisely the same spaces. To prove Theorem A we must show that \( \hat{V} \) and \( \hat{V} \) are PL homeomorphic to \( v_{n+1}(CP^1) \).

**Lemma 1.** \( \hat{V} \cong v_{n+1}(CP^1) \) if \( \sigma_2 \) is even.

We observe this from PL block bundle theory as follows: by construction \( \hat{V} \) is the union of two discs \( D^2_1 \times D^{2n} \) and \( c(\partial \hat{H}) = D^{2n+2} \) along \( S^1_1 \times D^{2n} \). Hence \( \hat{V} \) is trivially a block bundle regular neighborhood of \( CP^1 = D^2_1 \cup c(\partial D^{2n}) \). Assume the obstruction \( \sigma_2 \) is even. Then as noted by Sullivan ([23] p. 43) the splitting obstruction of the homotopy equivalence

\[
\tilde{h}: \hat{C}P^{n+1} \rightarrow CP^{n+1}
\]

along \( CP^1 \) vanishes as it is the mod 2 reduction of \( \sigma_2 \). Hence, by a homotopic deformation we may conclude that the transverse inverse image of \( CP^1 \) by \( \tilde{h} \) is \( CP^1 \subset \hat{C}P^{n+1} \). Moreover, as any two homotopic PL embeddings of \( CP^1 \subset \hat{C}P^{n+1} \) are ambiently PL isotopic (for \( n \geq 2 \) by Cor. 5.9 p. 65 [21]), we see by appeal to the uniqueness of normal block bundles (regular neighborhoods) [20] that \( \hat{V} \) is block bundle isomorphic to the bundle induced from \( v_{n+1}(CP^1) \) by \( \tilde{h} \). Conversely, the same argument on the homotopy inverse of \( \tilde{h} \) implies \( v_{n+1}(CP^1) \) is block bundle induced from \( \hat{V} \). As we are in the stable block and vector bundle range and \( \pi_2 B_{PL} = \pi_2 B_0 = Z_2 \) we can conclude that \( \hat{C} \) and \( v(CP^1) \) are block bundle isomorphic; hence PL homeomorphic.

**Lemma 2.** \( \hat{V} \cong S^2 \) (homotopy equivalent).

**Proof.** By construction \( \hat{V} = D^2 \times M_0^{2n} \cup X \cup c(\partial H) \) where \( X \) represents the solid handles we attached along \( S^1 \times M_0^{2n} \) to kill the homology kernel of \( \hat{h} \). The manifold \( D^2 \times M_0^{2n} \cup X \) is simply-connected with simply connected boundary and the homology of a point; hence by Smale's theorem (Thm. 1.1 [22]) it is a PL disc \( D^{2n+2} \). Thus, \( \hat{V} = D^{2n+2} \cup W D^{2n+2} \) where \( W \) is the complement of the embedding

\[
D^2 \times S^{2n-1} \subset S^{2n+1} = \partial D^{2n+2}
\]

and \( S^{2n-1} = \partial M_0^{2n} \). By the Mayer-Vietoris sequence we know that \( W \) is a homology circle. Then, by a second application of the Mayer-Vietoris sequence to the union \( D^{2n+2} \cup W D^{2n+2} \) we see that \( \hat{V} \) is a homology \( S^2 \). Finally, by the Van Kampen theorem \( \hat{V} \) is 1-connected and we apply the Whitehead theorem for CW complexes.
Lemma 3. $\hat{V} \cong v_{n+1}(CP^1)$.

Proof. $\partial \hat{V} = \partial[CP^{n+1} - \hat{V}] = \partial[CP^{n+1} - \hat{V}] = \partial \hat{V} \cong \partial v_{n+1}(CP^1)$ by Lemma 1. Let $S^2 \subset \hat{V}$ be a homotopy equivalence and a PL embedding via Whitney's embedding theorem. Then $S^2 \subset \hat{V} \subset \hat{CP}^{n+1}$ is homotopic to the standard embedding $CP^1 \subset \hat{CP}^{n+1}$, and as before, the PL block bundle neighborhoods of these two embeddings must be isomorphic. Let $v \subset \hat{V}$ be this block bundle. We note that $\partial v = \partial v_{n+1}(CP^1) \cong \partial \hat{V} = \partial \hat{V}$ by the previous lemmas. Hence, if $\hat{V} - v = Y$

we have $\partial Y = \partial \hat{V} \cup \partial v$, two copies of the same manifold.

We consider the Mayer-Vietoris sequence for the union $\hat{V} = Y \cup v$ over $\partial v = Y \cap v$:

$$\cdots \rightarrow H_1(\partial v)^{i_1 - i_2} \rightarrow H_1(v) \oplus H_q(Y)^{j_1 - j_2} \rightarrow H_1(\hat{V}) \rightarrow \cdots$$

where

$$i_1 : \partial v \ni v, \quad j_1 : v \ni \hat{V},$$

$$i_2 : \partial v \ni Y, \quad j_2 : Y \ni \hat{V}.$$ 

Since $v$ and $V$ are homotopy 2-spheres and $j_1$ is a homotopy equivalence, we see that for $q \neq 2$, $i_2^*: H_q(\partial v) \rightarrow H_q(Y)$ must be an isomorphism. When $q = 2$ the sequence becomes:

$$Z^{1-i_2} \rightarrow Z \oplus A \rightarrow Z, \quad A = H_2(Y)$$

from which we obtain $i_2^*$ are isomorphisms $Z \rightarrow A \rightarrow Z$. Thus, $i_2 : \partial v \subset Y$ is a homology isomorphism, and in fact, a homotopy equivalence since $\hat{V} = Y \cup v$ and $\hat{V}, v, \partial v$ are all 1-connected so that by Van Kampen's theorem $Y$ is 1-connected.

We show next that $\partial \hat{V} \subset Y$ is a homology isomorphism so that $Y$ is an $h$-cobordism from $\partial v$ to $\partial \hat{V}$—i.e. $Y \cong \partial v \times I$ and $\hat{V} = Y \cup v \cong v \cong \hat{V} v_{n+1}(CP^1)$ as required.
We know already that $\partial \hat{V} \cong Y$ as $\partial \hat{V} \cong \partial \nu \cong Y$. Moreover, $\partial \nu \cong \partial \nu_{n+1}(CP^1)$ is an $S^{2n-1}$ bundle over $S^2$. Hence, by the Serre Spectral Sequence we have

$$H_p(Y) = H_p(\partial \hat{V}) = \begin{cases} Z & \text{if } p = 0, 2, 2n - 1, 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the exact sequence of the pair $(\hat{V}, \partial \hat{V})$ is:

$$0 = H_3(\hat{V}, \partial \hat{V}) \rightarrow H_2(\partial \hat{V}) \rightarrow H_2(\hat{V}) \rightarrow H_1(\hat{V}, \partial \hat{V}) = 0$$

where the first and last groups are 0 by Poincaré Duality. Thus, the inclusion $\partial \hat{V} \subset Y \subset \hat{V}$ is a homology isomorphism through $p = 2$.

Now, consider the composition $f: \partial \hat{V} \rightarrow Y \rightarrow \partial \hat{V}$ where the second map is a homotopy equivalence. Then $f_*: H_p(\partial \hat{V}) \rightarrow H_p(\partial \hat{V})$ is an isomorphism for $p \leq 2$, and by Poincaré Duality so is $f^*: H^i(\partial \hat{V}) \rightarrow H^i(\partial \hat{V})$ for $q = 2n - 1, 2n, 2n + 1$. By the Universal Coefficient Theorem $f_*$ is an isomorphism for $p = 2n - 1, 2n, 2n + 1$ and so for all $p$. Thus, $f$ is a homotopy equivalence, and so is $i$.

Theorem A is now an immediate consequence of the last lemma as we have:

$$\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) = [CP^{n+1} - \hat{V}] \cup \hat{V},$$

$$\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0) = [CP^{n+1} - \nu_{n+1}(CP^1)] \cup_{\sigma_n} \nu_{n+1}(CP^1)$$

where we have identified $\hat{V}$ with $\nu_{n+1}(CP^1)$ by Lemma 1, and the PL homeomorphism

$$\varphi_n: \partial [\widetilde{CP}^{n+1} - \nu(CP^1)] \rightarrow \partial \nu(CP^1)$$

comes from the restriction to the boundary of the PL homeomorphism $\hat{V} \rightarrow \nu_{n+1}(CP^1)$ of Lemma 3.

III. Construction of the self-knotting $\varphi_n$: Here we construct for $n \geq 4$ a PL self-knotting

$$\varphi_n: \partial \nu_{n+1}(CP^1) \rightarrow \partial \nu_{n+1}(CP^1)$$

with the property that it extends to a homotopy equivalence

$$\overline{\varphi}_n: \nu_{n+1}(CP^1) \rightarrow \nu_{n+1}(CP^1)$$
which has a transverse-inverse image

\[ M_0^{2n} = \bar{q}_\sigma^{-1}(D^{2n}) \]
on a fiber \( D^{2n} \). Clearly such a \( \varphi_\sigma \) will suffice for the map in Theorem A.

We begin the construction by defining

\[ \Sigma^{2n-1}_\sigma \subset S^{2n+1} \]
to be the smooth Brieskorn knot represented as the link of the singularity on the hypersurface in \( C^{n+1} \) defined by

\[
p(Z) = \begin{cases} Z_0^{6a-1} + Z_1^2 + \cdots + Z_n^2, & n \text{ even}, \\ Z_0^2 + Z_1^2 + \cdots + Z_n^2, & n \text{ odd}. \end{cases}
\]

It is well-known that \( S^{2n+1} - \Sigma^{2n-1}_\sigma \) is a smooth fiber bundle over the circle with fiber \( M_0^{2n} \), the smooth Milnor or Kervaire manifold with surgery invariant \( \sigma \).

Now, let \( S^1 \subset S^{2n+1} \) be a fiber on the boundary of the smooth tubular neighborhood \( D^2 \times \Sigma^{2n-1}_\sigma \) of the knot (a trivial bundle as \( \pi_{2n-1}(\text{SO}(2)) = 0 \) for \( n > 1 \)). Since \( n > 1 \) this circle \( S^1 \) is smoothly unknotted in \( S^{2n+1} \) so that the complement of a small tube \( S^1 \times D^{2n} \) about it is diffeomorphic to \( D^2 \times S^{2n-1} \). Hence the knot \( \Sigma^{2n-1}_\sigma \) lies in this complement with a trivial normal bundle and we can therefore define:

\[ \beta: D^2 \times \Sigma^{2n-1}_\sigma \hookrightarrow D^2 \times S^{2n-1} \]
as this embedding. Let \( W^{2n+1} \) be the complement of this smooth embedding. Then we observe:

(a) \( \partial W = S^1 \times S^{2n-1} \cup S^1 \times \Sigma^{2n-1}_\sigma \).

(b) \( W \) is a smooth fiber bundle over the circle \( S^1 \) with fiber \( F^{2n} = M_0^{2n} - D^2 \) and \( \partial F = S^{2n-1} \cup \Sigma^{2n-1}_\sigma \).

(c) the bundle projection is trivial on \( \partial W \rightarrow S^1 \).

Now, using the smooth embedding \( \beta \) we define a piecewise-linear embedding

\[ \gamma_\sigma: D^2 \times S^{2n-1} \hookrightarrow D^2 \times S^{2n-1} \]
as the composite map

\[ D^2 \times S^{2n-1} \overset{\text{id} \times \alpha_\sigma}{\rightarrow} D^2 \times \Sigma^{2n-1}_\sigma \overset{\beta}{\rightarrow} D^2 \times S^{2n-1} \]
where \( \alpha_\sigma: S^{2n-1} \rightarrow \Sigma^{2n-1}_\sigma \) is a specific PL homeomorphism.
We now describe the normal bundle $\nu_{n+1}(CP^1)$ in $CP^{n+1}$ as:

$$\nu_{n+1}(CP^1) = D_+^2 \times S^{2n-1} \cup \rho D_+^2 \times S^{2n-1}$$

(*) where $\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$ is a smooth bundle automorphism representing an element in $\pi_1(SO(2n)) = Z/2Z$ ($n > 1$). [We note in fact that $\gamma_{n+1}(CP^1)$ is trivial for $n$ even and non-trivial for $n$ odd as it is the Whitney sum of $n$ copies of the canonical line bundle over $CP^1 = S^2$.]

In the above description we are expressing $CP^1$ as $S^2 = D_+^2 \cup D_+^2$. Using this representation we will define the self-knotting $\phi_\sigma$ by showing that the PL embedding

$$\gamma_\sigma: D_+^2 \times S^{2n-1} \to D_+^2 \times S^{2n-1}$$

may be extended to a PL homeomorphism on all of $V_{n+1}(CP^1)$. We will show this using the very agreeable bundle structure on the complement $W$ of the embedding $\gamma_\sigma$.

The map

$$\phi_\sigma: D_+^2 \times S^{2n-1} \cup \rho D_+^2 \times S^{2n-1} \to D_+^2 \times S^{2n-1} \cup \rho D_+^2 \times S^{2n-1}$$

will in fact be defined as the union of three maps —

1. $\gamma_\sigma: D_+^2 \times S^{2n-1} \to D_+^2 \times S^{2n-1}$,

2. $\eta: \tilde{W}^{2n+1} \to W^{2n+1}$,

3. $\text{id} \times \mu: D^2 \times \Sigma_{-\sigma}^{2n-1} \to D_+^2 \times S^{2n-1}$

where $\eta$ is a bundle homeomorphism of bundles over $S^1$ and $\mu: \Sigma_{-\sigma}^{2n-1} \to S^{2n-1}$ is a PL homeomorphism and

$$D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W}^{2n+1} = D_+^2 \times S^{2n+1}.$$

Essentially what we are producing in this construction is a map with the symmetric property that $\phi_\sigma$ embeds a fiber (the core of $D_+^2 \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma_{-\sigma}^{2n-1} \subset D_+^2 \times S^{2n-1}$ while $\phi_\sigma^{-1}$ embeds a fiber (the core of $D_-^2 \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma_{-\sigma}^{2n-1} \subset D_+^2 \times S^{2n-1}$.

The construction will be completed by (a) defining the bundle $\tilde{W}$ and the bundle map $\eta$ in (2), (b) showing that $D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W}$ is in fact $D_+^2 \times S^{2n-1}$ by a PL homeomorphism which is the identity on the boundary, (c) showing that the maps (1), (2), (3) agree on boundaries after taking the defining automorphism $\rho$ into account, and finally by (d) showing that $\phi_\sigma$ is homotopic to the identity.
We define the bundle $\tilde{W}$ over $S^1$ by defining its fiber $\tilde{F}$ and its monodromy map $\tilde{h}: \tilde{F} \to \tilde{F}$.

Recall that the $2n$-manifold $F$ (fiber of $W$) is $(n - 1)$ connected and that $\partial F = S^{2n-1} \cup \Sigma^{2n-1}$ where the smooth exotic sphere is defined as $\Sigma^{2n-1} = D^{2n-1}_+ \cup D^{2n+1}_-$ and $\sigma: S^{2n-2} \to S^{2n-2}$ is an exotic diffeomorphism.

Let $I \subset F$ be a path connecting the centers of the discs $D^{2n-1}_+$ and $D^{2n-1}_-$ of $\Sigma^{2n-1}$ and $S^{2n-1}$. Then a tubular neighborhood of $I$ is $I \times D^{2n-1}_+$. We define $\tilde{F}$ as the smooth manifold

$$\tilde{F} = [F - I \times D^{2n-1}_+] \cup [I \times D^{2n-1}_+]$$

where the union is taken over the diffeomorphism

$$\text{id} \times \sigma^{-1}: I \times S^{2n-2} \to I \times S^{2n-2}.$$ 

Then $\partial \tilde{F} = \Sigma^{2n-1} \cup S^{2n-1}$ as a smooth manifold and we can define a PL homeomorphism

$$\hat{\eta}: \tilde{F} \to F$$

where $\hat{\eta}$ is the identity on $F - I \times D^{2n-1}_+$ and is $\text{id}_I \times (\text{cone extension of } \sigma)$ on $I \times D^{2n-1}_+$.

Then we define the monodromy $\tilde{h}: \tilde{F} \to \tilde{F}$ as the composite map

$$\tilde{h} = \hat{\eta}^{-1} \circ h \circ \hat{\eta}$$

where $h: F \to F$ is the monodromy map defining the bundle $W$. Since $\partial W$ is a trivial bundle we know that $h$ is the identity map on $\partial F$. Hence, $\tilde{h}$ is the identity on $\partial \tilde{F}$ and the bundle $\tilde{W}$ has the trivial boundary

$$\partial \tilde{W} = S^1 \times \Sigma^{2n-1} \cup S^1 \times S^{2n-1}.$$ 

Since $\hat{\eta} \circ \tilde{h} = h \circ \hat{\eta}$ the PL homeomorphism $\hat{\eta}: \tilde{F} \to F$ induces a well-defined bundle homeomorphism

$$\eta: \tilde{W}^{2n+1} \to W^{2n+1}.$$ 

Restricted to the boundary $\eta$ is a pair of bundle maps

$$\text{id}_{S^1} \times \alpha^{-1}_{-\sigma}: S^1 \times \Sigma^{2n-1} \to S^1 \times S^{2n-1},$$

$$\text{id}_{S^1} \times \alpha_{\sigma}: S^1 \times S^{2n-1} \to S^1 \times \Sigma^{2n-1}$$

where the PL homeomorphism $\alpha_{-\sigma}$ and $\alpha_{\sigma}$ are the identity on $D^{2n-1}_-$ and the cone extension of $\sigma^{-1}$ and $\sigma$ respectively on $D^{2n-1}_+$. 

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We next embed \( \tilde{W} \) in \( D^2 \times S^{2n-1} \) as a knot complement which will act as an inverse to \( W \):

Recall the bundle isomorphism

\[
\rho: S^1 \times S^{2n-1} \rightarrow S^1 \times S^{2n-1}
\]

which defines \( \partial_{\nu_{n+1}}(CP^1) \). We define a PL bundle map

\[
\hat{\rho}: S^1 \times \Sigma_{-\sigma}^{2n-1} \rightarrow S^1 \times \Sigma_{-\sigma}^{2n-1}
\]

as the composite: \( \hat{\rho} = (id_{S^1} \times \alpha_{-\sigma}) \cdot \rho \cdot (id_{S^1} \times \alpha_{-\sigma})^{-1} \). We consider the PL manifold

\[
D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1}
\]

where the union is over the appropriate component of \( \partial \tilde{W} \) and show:

**Proposition.** The PL manifold \( D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1} \) is isomorphic to \( D^2 \times S^{2n-1} \) by a PL homeomorphism \( \Delta \) which restricted to the boundary \( S^1 \times S^{2n-1} \) is an \( S^{2n-1} \) bundle isomorphism \( \lambda \).

**Proof.** We recall from the definition of \( W^{2n+1} \) that \( S^1 \times D^{2n} \cup W^{2n+1} \) is the knot complement of our original Brieskorn knot and so has the homology of \( S^1 \). A simple exercise with the Mayer-Vietoris sequence implies then that the manifold \( \tilde{W}^{2n+1} \cup S^1 \times D^{2n} \) likewise is a homology circle, and a second application of the sequence implies that the PL manifold.

\[
P^{2n+1} = D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W} \cup S^1 \times D^{2n}
\]

has the homology of \( S^{2n+1} \). Moreover, \( P^{2n+1} \) is simply connected since \( \tilde{W} \cup S^1 \times D^{2n} \) fibers over \( S^1 \) with fiber \( \tilde{F}^{2n} \cup D^{2n} \) which is \( (n - 1) \)-connected. Hence \( \pi_i(\tilde{W} \cup S^1 \times D^{2n}) = Z \) and by the Van Kampen theorem on the union

\[
[D^2 \times \Sigma_{-\sigma}^{2n-1}] \cup_{S^1 \times \Sigma_{-\sigma}} [\tilde{W} \cup S^1 \times D^{2n}]
\]

we have \( \pi_i(P^{2n+1}) = 0 \). By the Hurewicz and Whitehead theorems any simply-connected homology sphere is a homotopy sphere, and by the generalized Poincaré conjecture \((2n + 1 \geq 9)P^{2n+1} \) is a PL sphere.

The identification \( D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W}/S^1 \times D^{2n} \cong S^{2n+1} \) provides a PL embedding \( S^1 \subset S^{2n+1} \) and exhibits \( i(S^1 \times D^{2n}) \subset S^{2n+1} \) as a representative for the PL normal microbundle to this embedding. We apply a
theorem due to Lashof and Rothenberg (Thm. 7.3 in [13]) to obtain a 
piecewise differentiable homeomorphism \( g: S^{2n+1} \to S^{2n+1} \) so that \( g \circ i: S^1 \times D^{2n} \to S^{2n+1} \) is the smooth vector bundle to the smooth embedding \( g \circ i: S^1 \to S^{2n+1} \). By smoothly unknotting this circle and applying the 
smooth tubular neighborhood theorem we obtain a diffeomorphism \( h: S^{2n+1} \to S^{2n+1} \) so that

\[
\begin{align*}
\lambda \downarrow & \uparrow j \\
h \circ g \circ i: S^1 \times D^{2n} & \to S^{2n+1} \\
S^1 \times D^{2n} & \to S^{2n+1}
\end{align*}
\]

commutes where \( j \) is the standard embedding and \( \bar{\lambda} \) is a vector bundle isomorphism. Hence, the restriction map

\[
h \circ g \mid: S^{2n+1} - i(S^1 \times D^{2n}) \to S^{2n+1} - j(S^1 \times D^{2n})
\]

defines a piecewise differentiable homeomorphism

\[
\Lambda: \left[D^2 \times S^{2n-1}_\sigma \cup \bar{\lambda} \mathrm{W} \right] \to D^2 \times S^{2n-1}
\]

which restricts as \( \lambda = \bar{\lambda} \) on the boundary. Finally, we observe that (cf. 
Cor. 10.13 in [19]) we may choose a smooth triangulation of \( D^2 \times S^{2n-1} \) 
so that \( \Lambda \) is PL. Now, using the homeomorphisms \( \Lambda \) and \( \eta \) we define a PL homeomorphism:

(1) \( \varphi_{\sigma}: \xi \to \partial r_{n+1}(CP^1) \)

where \( \xi \) is the \( S^{2n-1} \) bundle over \( CP^1 = S^2 \) defined by \( \lambda^{-1}. \):

\[
\xi = D^2_\sigma \times S^{2n-1} \cup \lambda_\sigma, D^2_+ \times S^{2n-1}
\]

\[
\Lambda^{-1} \cup \id \to D^2 \times S^{2n-1}_\sigma \cup \rho \mathrm{W}^2 \cup \id D^2_+ \times S^{2n-1}
\]

\[
(id \times \alpha_\sigma) \cup \eta \cup (id \times \alpha_\sigma) \to D^2_\sigma \times S^{2n-1} \cup \rho \mathrm{W} \cup D^2 \times S^{2n-1}
\]

\[
= D^2_\sigma \times S^{2n-1} \cup \rho D^2_+ \times S^{2n-1} = \partial r_{n+1}(CP^1)
\]

From the next lemma to the effect that two non-isomorphic sphere 
bundles over \( S^2 \) cannot be PL homeomorphic it follows that the existence 
of the map \( \varphi_{\sigma} \) itself guarantees that \( \xi \) and \( \partial r_{n+1}(CP^1) \) are the same bundle.

**Lemma.** For \( m \geq 3 \) the unique non-trivial orthogonal \( S^m \) bundle over \( S^2 \), 
\( \xi \), is not PL homeomorphic to \( S^2 \times S^m \).
Proof. Suppose \( t: \xi \to S^2 \times S^m \) is a PL homeomorphism. Let \( E \) be the non-trivial \( D^{m+1} \) bundle over \( S^2 \) with \( \partial E = \xi \) and define the PL manifold

\[ M^{m+3} = E \cup_j D^3 \times S^m \]

\( M \) is the union of simply connected spaces over a path connected intersection. Hence, \( \pi_1(M) = \{1\} \). For \( m \geq 3 \) the homotopy exact sequence of the fibration \( S^m \to \partial E \to S^2 \) implies that \( p_*: \pi_2(\partial E) \to \pi_2(S^2) \) is an isomorphism, and by the Whitehead theorem so is the inclusion \( H_2(\partial E) \to H_2(E) \). Hence, in the Mayer-Vietoris sequence

\[ \cdots \to H_j(S^2 \times S^m) \xrightarrow{\psi_j} H_j(E) \oplus H_j(D^3 \times S^m) \to H_j(M) \to \cdots \]

\( \psi_j \) is an isomorphism for \( j \leq m + 1 \). Trivially, \( H_{m+2}(M) = 0 \), and again we have an \((m+2)\)-connected \((m+3)\)-dimensional PL manifold which is consequently a PL sphere.

Then, \( E \cup_j D^3 \times S^m = S^{m+3} \) defines the vector bundle \( E \) as a PL normal micro-bundle to the embedding of its zero section \( S^2 \hookrightarrow S^{m+3} \). By Zeeman's PL unknotting theorem and the uniqueness [7] of stable PL normal microbundles, we see that \( E \) and \( S^2 \times D^{m+1} \) must be micro-bundle isomorphic. Let \( S^2 \to BO \to BPL \) be trivial, and as by smoothing theory the fiber \( PL/0 \) is 6-connected we see that \( b \) is homotopically trivial. As \( E \) was assumed non-trivial as a vector bundle the PL homeomorphism \( t \) cannot exist.

Thus, we define

\[ \varphi_a: \partial v_{n+1}(CP^1) = \xi \to \partial v_{n+1}(CP^1) \] from (1) as required.

Next we show that the \( \varphi_a \) just constructed is indeed a self-knotting and that it will suffice for Theorem A.

Recalling from bundle theory that every \( S^N \) bundle over \( S^2 \) for \( N \geq 2 \) has a section, we show

**Proposition.** Any orientation preserving PL homeomorphism \( \varphi: v \to v \), \( v \) an orthogonal \( S^N \) bundle over \( S^2 \), which embeds a section \( S^2 \hookrightarrow v \) homotopically to itself is homotopic to the identity.

**Proof.** A tubular neighborhood of the section \( j(S^2) \) is a \( D^N \) bundle \( U \) in the same stable bundle class as \( v \). \( \varphi(U) \) PL embeds this bundle in \( v \) with an inherited smooth structure. By the main theorem of smoothing
theory ([8] or [13], Thm. 7.3) and the uniqueness of smoothings on $S^2$ we can piecewise differentially isotope this embedding to a smooth embedding of $U \to \nu$. We may easily make the isotopy ambient. Next, we smoothly unknot the core sphere of $U$ and apply the smooth tubular neighborhood theorem. We have, therefore, P.D. isotoped $\varphi$ so that restricted to $U$ it is a $D^N$ bundle isomorphism. Since $\pi_2(\text{SO}(N)) = 0$ we can isotope this bundle mapping to the identity through bundle isomorphisms on $U$ all of which extend to $\nu$ as $U$ is a sub-bundle. Thus, we have isotoped $\varphi$ so that it is the identity on $U$. Now, $\nu - U \cong U$ as each fiber of $U$ is a hemisphere of a fiber in $\nu$. We isotope $\varphi_{rel(U)}$ so that it is the identity on the zero section of the bundle $\nu - U$. Finally, we homotope $\varphi$ to the identity by collapsing the fibers of $\nu - U$ to the zero-section.

We observe that the $\varphi_\sigma$ constructed above satisfies the hypothesis of this last proposition as follows: $\varphi_\sigma$ is orientation preserving by construction. Also, as the original Brieskorn knot embedded a fiber $S^{2n+1}$ homotopically to the usual embedding, we know that $\varphi_\sigma$ does also. That is $(\varphi_\sigma)_\#(\partial\nu) = [\partial\nu]$ and $(\varphi_\sigma)^*(e^{2n-1}) = e^{2n-1}$, where $e^{2n-1} \in H^{2n-1}(\partial\nu)$ is the class represented by inclusion of a fiber. By Poincaré Duality, then, $(\varphi_\sigma)_\#(e_2) = e_2$ for $e_2 \in H_2(\partial\nu)$ the class dual to $e^{2n-1}$. This implies by the Hurewicz Theorem that $\varphi_\sigma$ induces the identity homomorphism on $\pi_2(\partial\nu)$, which is generated by the inclusion of a section.

The map $\varphi_\sigma$ constructed in section C embeds a fiber $S^{2n-1}$ onto the image of the Brieskorn knot. Hence, in the decomposition
$$\hat{\mathbb{C}P}^n = [\mathbb{C}P^n - \nu_{n+1}(\mathbb{C}P^1)] \cup_{\varphi_\sigma} [\nu_{n+1}(\mathbb{C}P^1)]$$
the identification is in the order:
$$\varphi_\sigma : \partial[\mathbb{C}P^n - \nu] \to \partial\nu.$$ 
To show, therefore, that $\hat{\mathbb{C}P}^n \leftrightarrow (0, \ldots, 0, \sigma)$ we must extend $\varphi^{-1}_\sigma$ to a homotopy equivalence $\varphi^{-1}_\sigma : \nu \to \nu$ with transverse-inverse image of a fiber being the Milnor or Kervaire manifold $M^{2n}_0$. Note that any extension will be a homotopy equivalence as $\nu \cong S^2$ and $\varphi^{-1}_\sigma$ induces the identity on $\pi_2(\partial\nu) = \pi_2(\nu)$.

**Proposition.** The PL homeomorphism $\varphi^{-1}_\sigma : \partial\nu_{n+1}(\mathbb{C}P^1) \to \partial\nu_{n+1}(\mathbb{C}P^1)$ constructed above extends to $\varphi^{-1}_\sigma : \nu_{n+1}(\mathbb{C}P^1) \to \nu_{n+1}(\mathbb{C}P^1)$ with transverse-inverse image
$$(\varphi^{-1}_\sigma)^{-1}(D^{2n}) = M^{2n}_0$$
Proof. \((\varphi^{-1}_o)^{-1}(S^{2n-1}) = \varphi_o(S^{2n-1}) = \Sigma^{2n-1}_o \subset \partial \nu\) by the construction of \(\varphi_o\). Moreover, the restriction \(\varphi^{-1}_o|: D^2 \times \Sigma^{2n-1}_o \to D^2_+ \times S^{2n-1}\) is a product map. Now, \(\Sigma^{2n-1}_o\) bounds a fiber \(F^{2n} \subset W^{2n+1}\) whose other boundary component is a fiber \(S^{2n-1}\) of \(\partial \nu\). Let \(D^{2n} \subset \nu\) be the fiber whose boundary is this same sphere. Then, \(F^{2n} \cup D^{2n} = M^{2n}_o\) by the definition of \(F^{2n}\). By pushing \(F^{2n}\) into \(\nu\) along a vector field normal to \(\partial \nu\) and smoothing the corner at \(S^{2n-1}\) between \(F^{2n}\) and \(D^{2n}\) we obtain a smooth embedding \(M^{2n}_o \hookrightarrow \nu\) extending
\[
\partial M^{2n}_o = \Sigma^{2n-1}_o \subset \partial \nu.
\]
Moreover, this embedding will have trivial normal \(D^2\) bundle as \(H^1(M^{2n}_o, \mathbb{Z}) = 0\). Hence, we can extend the product map
\[
\varphi^{-1}_o: D^2 \times \Sigma^{2n-1}_o \to D^2_+ \times S^{2n-1}
\]
to a bundle map \(\hat{\varphi}^{-1}_o: D^2 \times M^{2n}_o \to D^2_+ \times D^{2n}\) covering a degree one extension \(M^{2n}_o \to D^{2n}\). Since \([\nu - D^2_+] \times D^2_- \times D^{2n} = D^{2n-2}\) there are no cohomology obstructions to extending
\[
\varphi^{-1}_o \cup \hat{\varphi}^{-1}_o \text{ to } \varphi^{-1}_o: \nu \to \nu
\]
with the required transverse-inverse image built in.

References


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