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Graphs with k-Uniform Edge Betweenness Centrality

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by

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Abstract

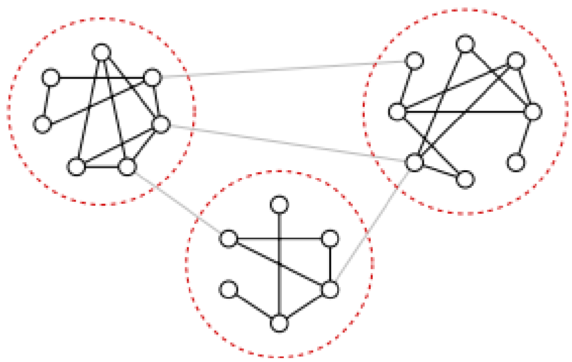
The *edge betweenness centrality* of an edge is defined as the ratio of shortest paths between all pairs of vertices passing through that edge. A graph is said to have k-uniform edge betweenness centrality, if there are k different edge betweenness centrality values. In this thesis we investigate graphs that have k-uniform edge betweenness centrality where $k = 2$ or 3, and precisely determine the edge betweenness centrality values for various families of graphs.

1 Introduction

The *betweenness centrality* of a vertex v is the ratio of the number of shortest paths between two other vertices u and w which contain v to the total number of shortest paths between u and w, summed over all ordered pairs of vertices (u, w). This idea was introduced by Anthonisse [2] and Freeman [7] in the context of social networks, and has since appeared frequently in both social network and neuroscience literature [8, 21, 5, 9, 13, 12, 10, 6, 16, 23]. Here are some definitions that will be used later on. We note that, for undirected graphs, shortest paths from x to y are regarded as the same as shortest paths from y to x, though the associated contribution to the sum $B'_G(e)$ is double-counted. Many results on edge betweenness were obtained by Gago [10, 12], and [13]. The case where $k = 1$ was studied by Newman, Miranda, Flórez, and Narayan [19].

2 Background

Consider points, which we will call vertices, interconnected with one another with the use of some edges.



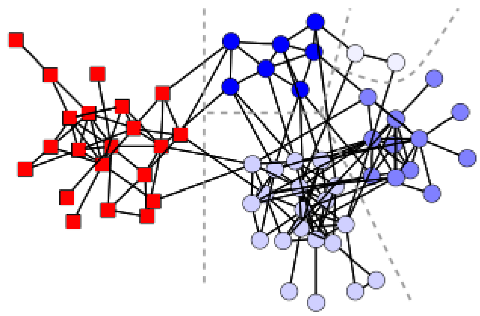
We can see that there are some edges which are more important to the graph as a whole than others. Some edges are considered more important because if we, for example, take those important edges away; we will be left with a set of a few distinct clusters of vertices which we will call communities.

When separating a graph into communities we use the *Girvan-Newman Algorithm*. This method says:

- Calculate all edge betweenness centralities
- Remove the edge with the highest edge betweenness centrality
- Recalculate all edge betweenness centralities
- Repeat step 2 until there are no edges left

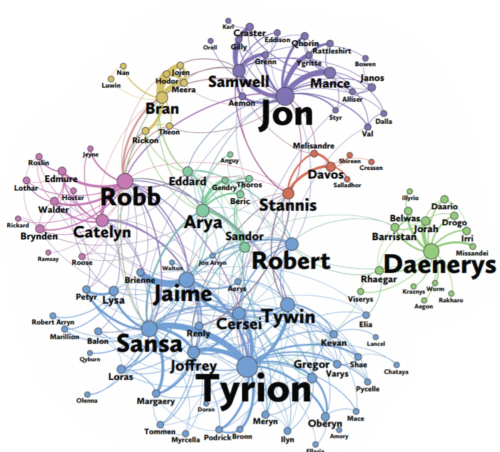
We will be studying graphs that have k-uniform edge betweenness centrality, meaning that the graphs have exactly k edge betweenness centrality values. In the case where k=1, when using the Girvan-Newman Algorithm the choice of the first edge to remove can be done arbitrarily (no edge is better than any other edge). In general, for graphs with k-uniform edge betweenness centrality there is a choice between k types of edges to remove in the first step.

Now consider a more intricate graph: We can see that this graph is separated into five communities. The



separation is based on the Girvan-Newman Algorithm. This is actually a graph of a social network of bottlenose dolphins. Each node is a dolphin with the circles and squares as the primary separation and the different colored circles as a secondary separation.

Consider the following graph where the vertices are the ones being weighed. As we can see here, this is a graph



of relationships within the popular TV show, *Game of Thrones*. As the larger nodes represent the vertices with the biggest betweenness centrality. Similarly as with the edges, the Girvan-Newman Algorithm would work here as well.

3 Definitions

Definition 3.1. *The* ***edge betweenness centrality*** *of an edge e in a graph G, denoted $B'_G(e)$ (or simply $B'(e)$ when G is clear), measures the frequency at which e appears on a shortest path between two distinct vertices x and y. Let σ_{xy} be the number of shortest paths between distinct vertices x and y, and let $\sigma_{xy}(e)$ be the number of shortest paths between x and y that contain e. Then $B'_G(e) = \sum_{x,y} \frac{\sigma_{xy}(e)}{\sigma_{xy}}$ (for all distinct vertices x and y). We will also define the total edge betweenness centrality of a graph to be $B'(G)$.*

Definition 3.2. *A graph G has* ***k-uniform edge betweenness centrality****, or is* ***edge-betweenness-k-uniform****, if $B'_G(e)$ has exactly k values over all edges e in G.*

Definition 3.3. *A* ***circulant graph*** *$C_n(L)$ is a graph on vertices $v_1, v_2, ..., v_n$ where each v_i is adjacent to $v_{(i+j) \pmod n}$ and $v_{(i-j) \pmod n}$ for each j in a list L.*

Definition 3.4. *A* ***diametrical edge*** *is an edge on a circulant graph $C_n(L)$ between two vertices $v_i - v_{i+\frac{n}{2}}$ with the vertex number being $\pmod n$. This edge, in effect, cuts the graph in half.*

4 Results

For our research, we have looked at different circulant graphs and their properties. First, consider a circulant graph on n vertices with edges around the perimeter as well as diametrical edges throughout. To have a diametrical chord, the value of n will have to be even. We will first consider the cases where n is a multiple of 4. We will look at the diametrical chord.

Lemma 1: *On a circulant graph of the form $C_n(1, \frac{n}{2})$, making a path from v_1, we can traverse to v_i, where $2 \leq i \leq k$ and $3k \leq i \leq 4k$, strictly using the outer perimeter.*

Proof. We have a graph with $4k$ vertices with $4k$ edges around the perimeter. Separate the graph into quarters such that the dividing vertices end up being v_1, v_k, $v_{\frac{4k}{2}}$, and v_{3k}. We are able to say that these vertices will always exist and will divide the graph as stated because the total number of vertices is $4k$ and $\frac{4k}{4} = k$ which is the number or vertices in each quarter. If we wish to traverse from v_1 to any vertex v_i where $i < k$, we can easily go around the perimeter and have the shortest edge without needing to traverse a diametrical chord. Say we wish to find the shortest path between v_1 and v_k. We can see that v_1 and $v_{\frac{4k}{2}}$ are equidistant from v_k. This means traversing the diametrical chord would be an extra step, adding an edge to the path. Thus, to find the shortest path would mean not traversing the diametrical chord and instead traversing the perimeter. The same will hold for a path between v_1 and v_{3k}. Hence, it will always be true that the shortest paths between v_1 and v_i, where $2 \leq i \leq k$ and $3k \leq i \leq 4k$, can always be traversed with a path exclusively on the perimeter. ∎

Theorem 2: *If you have a circulant graph $C_n(1, \frac{n}{2})$ where $n = 4k$ edges, the edge betweenness centrality of the diametrical chord will be $n - 2$.*

Proof. We will proceed by proof by induction. Consider a circulant graph with 8 vertices of the form $C_8(1, 4)$. We will number the vertices counter-clockwise. Consider the diameter edge of $v_1 - v_5$.

There is only one way to get a chord from $v_1 - v_5$ and it is through the colored edge so it will have an edge betweenness centrality of $B'_1(e) = 1$. From $v_1 - v_4$ there are two paths with one of them going through the colored edge:

$$v_1 - v_5 - v_4$$

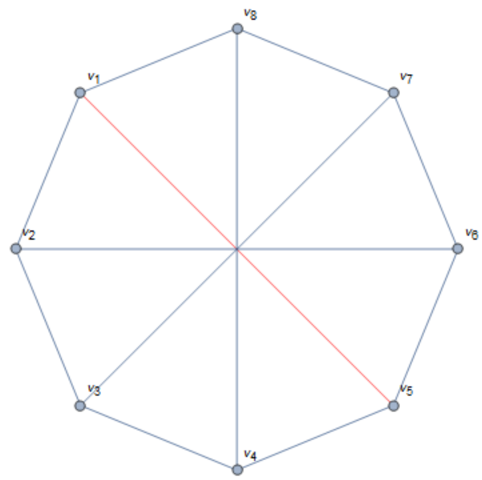


Figure 1: $C_8(1,4)$

$$v_1 - v_8 - v_4$$

This would make the edge betweenness centrality for $v_1 - v_4$ to be $\frac{1}{2}$. There are three such other cases from $v_1 - v_6$, $v_8 - v_5$, and $v_2 - v_5$. So the total edge betweenness centrality on two edges is $B_2'(e) = 4(\frac{1}{2}) = 2$. Consider shortest paths on three edges. As we can see by the graph, any path between two vertices on three edges can be traversed by two edges from the other direction. Thus there are no shortest paths of length 3. If we add the total centralities and multiply it by 2 to take into consideration the backwards path, we get $2(1+2) = 6$ and we notice $6 = 8 - 2$.

Now consider $n = 4k$ where k is a positive integer. The circulant graph will be $C_{4k}(1, \frac{4k}{2})$. Imagine an edge from v_1 to $v_{\frac{4k}{2}+1}$. There will be only one path from v_1 to $v_{\frac{4k}{2}+1}$. So the edge betweenness centrality for paths on one edge is $B_1'(e) = 1$. We will notice that $a - b = 1 - \frac{4k}{2} + 1 \equiv \frac{4k}{2} \pmod{4k}$. Now look at shortest paths with two edges. There will be a total of 8 paths between 4 different pairs of vertices:

- $v_1 - v_{\frac{4k}{2}}$:

$$v_1 - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}}$$

$$v_1 - v_{4k} - v_{\frac{4k}{2}}$$

- $v_1 - v_{\frac{4k}{2}+2}$:

$$v_1 - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}+2}$$

$$v_1 - v_2 - v_{\frac{4k}{2}+2}$$

- $v_2 - v_{\frac{4k}{2}+1}$:

$$v_2 - v_1 - v_{\frac{4k}{2}+1}$$

$$v_2 - v_{\frac{4k}{2}+2} - v_{\frac{4k}{2}+1}$$

- $v_{4k} - v_{\frac{4k}{2}+1}$:

$$v_{4k} - v_1 - v_{\frac{4k}{2}+1}$$

$$v_{4k} - v_{\frac{4k}{2}} - v_{\frac{4k}{2}+1}$$

Their total edge betweenness centrality will be $B_2'(e) = 4(\frac{1}{2}) = 2$. We notice that between two points v_1 and $v_{\frac{4k}{2}}$ we have $1 - \frac{4k}{2} \equiv \frac{4k}{2} - 1 \pmod{4k}$. Now look at shortest paths between two points on three edges. There will be a total of 18 paths between 6 pairs of vertices.

- $v_1 - v_{\frac{4k}{2}-1}$:

$$v_1 - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}} - v_{\frac{4k}{2}-1}$$

$$v_1 - v_{4k} - v_{\frac{4k}{2}} - v_{\frac{4k}{2}-1}$$

$$v_1 - v_{4k} - v_{4k-1} - v_{\frac{4k}{2}-1}$$

- $v_1 - v_{\frac{4k}{2}+3}$:

$$v_1 - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}+2} - v_{\frac{4k}{2}+3}$$

$$v_1 - v_2 - v_{\frac{4k}{2}+2} - v_{\frac{4k}{2}+3}$$

$$v_1 - v_2 - v_3 - v_{\frac{4k}{2}+3}$$

- $v_2 - v_{\frac{4k}{2}}$:

$$v_2 - v_{\frac{4k}{2}+2} - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}}$$

$$v_2 - v_1 - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}}$$

$$v_2 - v_1 - v_{4k} - v_{\frac{4k}{2}}$$

- $v_3 - v_{\frac{4k}{2}+1}$:

$$v_3 - v_{\frac{4k}{2}+3} - v_{\frac{4k}{2}+2} - v_{\frac{4k}{2}+1}$$

$$v_3 - v_2 - v_{\frac{4k}{2}+2} - v_{\frac{4k}{2}+1}$$

$$v_3 - v_2 - v_1 - v_{\frac{4k}{2}+1}$$

- $v_{4k} - v_{\frac{4k}{2}+2}$:

$$v_{4k} - v_{\frac{4k}{2}} - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}+2}$$

$$v_{4k} - v_1 - v_{\frac{4k}{2}+1} - v_{\frac{4k}{2}+2}$$

$$v_{4k} - v_1 - v_2 - v_{\frac{4k}{2}+2}$$

- $v_{4k-1} - v_{\frac{4k}{2}+1}$:

$$v_{4k-1} - v_{\frac{4k}{2}-1} - v_{\frac{4k}{2}} - v_{\frac{4k}{2}+1}$$

$$v_{4k-1} - v_{4k} - v_{\frac{4k}{2}} - v_{\frac{4k}{2}+1}$$

$$v_{4k-1} - v_{4k} - v_1 - v_{\frac{4k}{2}+1}$$

We will see that there are three edges between each set of two vertices and only one of the edges will go through the colored edge between v_1 to $v_{\frac{4k}{2}+1}$. Looking at each set of three edges, we can see that the path from v_1 to $v_{\frac{4k}{2}+1}$ appears in only one of the three edges for each pair of vertices. We can also see that for each vertex pair, the edge v_1 to $v_{\frac{4k}{2}+1}$ appears as either the first connecting edge, the second connecting edge, or the third. We also note that to get across the graph, one would need to traverse a single diametrical chord. Since there is only one diametrical chord that needs crossing for each set of edges and there are three different edges between the vertex pairs, the colored edge can only be traversed as one of the three edges crossed. This would then make $B_3'(e) = 6(\frac{1}{3}) = 2$. Here we notice $1 - \frac{4k}{2} - 1 \equiv \frac{4k}{2} - 2 \pmod{4k}$. For a circulant graph on $4k$ vertices, there will be shortest paths between vertices on $1, 2, ..., \frac{4k}{4} = k$ edges. This is because moving from vertex labeled v_1 counter-clockwise, we will get $\frac{1}{4}$ around the circle until we reach a vertex where there is a shorter way to get to that extra vertex. We also notice, that as we increase the number of edges in the path, we decrease the modulus that the difference of the two vertices is equal to by 1 as well. We will also find that total edge betweenness centrality, accounting for paths from a to b and b to a, we have $B'(e) = 2(1 + 2(k-1))$.

As we can see, this does follow the pattern that $B'(e) = n-2$. Hence, by induction, this will hold for all k. ■

Now we are going to consider the short outer chord on the same graph.

<u>Lemma 3:</u> *On a circulant graph of the form $C_n(1, \frac{n}{2})$, there will be paths with at most k edges.*

Proof. Consider a graph on $4k$ vertices with $4k$ edges around the perimeter. We have $k = \frac{n}{4}$. Consider a path on $k+1$ edges. Since from $\frac{n}{2}$ there are k edges to the left and k edges to the right, trying to traverse $k+1$ edges would not be the shortest path. It would be shorter to traverse k edges from v_1 in either direction than to traverse $v_1 - v_{\frac{n}{2}}$ to the desired vertex. So the maximum length of shortest paths would end up being $k = \frac{n}{4}$. For this reason, the vertices v_{3k+1} and v_{k+2} are not traversed along with the colored edge since there is a shorter path which does not contain it. ■

Now consider the paths between two vertices on k edges. The vertex pairs we will traverse are as follows:

$$v_1 - v_{3k}$$

$$v_{4k} - v_{3k-1}$$

$$v_{4k-1} - v_{3k-2}$$

$$\vdots$$

$$v_{3k+2} - v_{2k+2}$$

There will be a total of $2(k-1)$ different vertices with $k-1$ different betweenness centralities.

<u>Lemma 4:</u> On a circulant graph of the form $C_n(1, \frac{n}{2})$, there will be $2(m-1)$ different vertex paths with m vertices and there will be $m-1$ different betweenness centralities.

Proof. If we have a path with m vertices, we will have $m-1$ edges between them. We know that the edge $v_1 - v_2$ will appear in one of $m-1$ different locations. As the edge $v_1 - v_2$ moves from the beginning to the end of the path, the edge betweenness centrality decreases. This method will give the first $m-1$ distinct centralities which start from vertices v_1, v_{4k}, v_{4k-1}, $\cdots$,v_{3k+2}. The other $m-1$ vertices behave in an opposite manner from the first $m-1$. These paths start from v_{2k+1}, v_{2k}, $\cdots$,v_{k+3}. These paths then end on vertices v_k, v_{k-1}, $\cdots$, v_2. As we can see, as the edge $v_1 - v_2$ moves from the beginning to the end of the path, the edge betweenness centrality increases unlike stated previously. These paths also do not double count. Hence, we will have $2(m-1)$ different paths with $m-1$ different betweenness centralities. ■

Theorem 5: *If you have a circulant graph $C_n(1, \frac{n}{2})$ where $n = 4k$ edges with $k \in \mathbb{Z}$, the edge betweenness centrality of the short chord will be $2k^2$.*

Proof. We will proceed by proof by induction. Consider a circulant graph with 12 vertices of the form $C_{12}(1,6)$.We will number the vertices counter-clockwise. consider the chord $v_1 - v_2$.

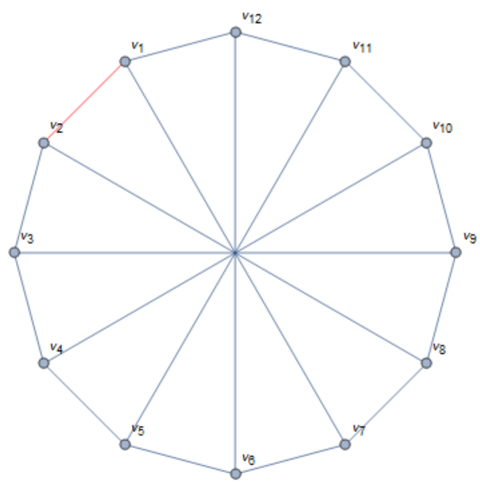


Figure 2: $C_{12}(1,6)$

We have that there is only one way to traverse each of the following paths and all of them go through the colored edge:

$$v_1 - v_2$$

$$v_1 - v_2 - v_3$$

$$v_{12} - v_1 - v_2$$

$$v_1 - v_2 - v_3 - v_4$$

$$v_{12} - v_1 - v_2 - v_3$$

$$v_{11} - v_{12} - v_1 - v_2$$

Each of these will give us a edge betweenness centrality of 1 totaling to $B'_1(e) = 6$. Now look at the path $v_1 - v_8$. There are two ways to traverse it with one of them going through the colored edge:

$$v_1 - v_2 - v_8$$

$$v_1 - v_7 - v_8$$

This means the edge betweenness centrality of $v_1 - v_8$ is $\frac{1}{2}$. There is one other such path from $v_7 - v_2$. So the total edge betweenness centrality on two edges is $B'_2(e) = 2\left(\frac{1}{2}\right) = 1$. Now look on the paths on three edges. We will have the four following paths:

$$v_1 - v_2 - v_3 - v_9, v_1 - v_2 - v_8 - v_9, v_1 - v_7 - v_8 - v_9$$

$$v_{12} - v_1 - v_2 - v_8, v_{12} - v_1 - v_7 - v_8, v_{12} - v_6 - v_7 - v_8$$

$$v_7 - v_1 - v_2 - v_3, v_7 - v_8 - v_2 - v_3, v_7 - v_8 - v_9 - v_3$$

$$v_6 - v_{12} - v_1 - v_2, v_6 - v_7 - v_1 - v_2, v_6 - v_7 - v_8 - v_2$$

We can see that in each set of paths, there are three different ways to get between each set of two vertices. We can also see that the first and last path sets have $B'(e) = \frac{2}{3}$ and the second and third path sets have $bc = \frac{1}{3}$. So the total edge betweenness centrality on three edges is $B'_3(e) = 2\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right) = 2$. Now if we total the betweenness centralities we found and multiply that value by two to account for reverse paths, we get the total edge betweenness centrality of $C_{12}(1,6)$ to be $B'(e) = 2(6+1+2) = 18$.

If we look at $C_{16}(1,8)$ we find that

$$B'_1(e) = 10, B'_2(e) = 1, B'_3(e) = 2, \text{ and } B'_4(e) = 3$$

$$B'(e) = 2(10+1+2+3) = 2(16) = 2(4^2) = 2k^2 = 32$$

Then looking at $C_{20}(1,10)$ we find

$$B'_1(e) = 15, B'_2(e) = 1, B'_3(e) = 2, B'_4(e) = 3, \text{ and } B'_5(e) = 4$$

$$B'(e) = 2(15+1+2+3+4) = 2(25) = 2(5^2) = 2k^2 = 50$$

From here consider $C_n(1, \frac{n}{2})$ where $n = 4k$ edges with $k \in \mathbb{Z}$. For $B_1'(e)$ we will have triangular numbers such that $B_1'(e) = \binom{k+1}{2}$. Now consider the paths that only have 1 option for traversing. We have the following:

$$v_1 - v_2$$

$$v_1 - v_2 - v_3$$

$$\vdots$$

$$v_1 - \cdots - v_{k+1}$$

$$v_{4k} - v_1 - v_2$$

$$v_{4k} - v_1 - v_2 - v_3$$

$$\vdots$$

$$v_{4k} - \cdots - v_k$$

$$v_{4k-1} - v_{4k} - v_1 - v_2$$

$$\vdots$$

$$v_{4k-1} - \cdots - v_{k-1}$$

$$\vdots$$

$$v_{3k+2} - \cdots - v_2$$

The way these paths are moving we see that it is like rotating the entire circle. Imagine the triangular numbers. First there is 1 dot which represents the path $v_1 - v_2$ making the edge betweenness centrality $\binom{2}{2} = 1$. Then the next level has 2 dots which represent 1-2-3 and $4k$-1-2 making the edge betweenness centrality $1+2 = \binom{3}{2} = 3$. This pattern will continue and the total edge betweenness centrality with only 1 path will be $B_1'(e) = \binom{k+1}{2}$. Now consider the edge betweenness centrality on paths between two vertices on two edges which use the diametrical chord. Consider the path from v_1 to $v_{\frac{n}{2}+2}$. That path can be traversed in two ways:

$$v_1 - v_2 - v_{\frac{n}{2}+2}$$

$$v_1 - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2}$$

We can see that there are two different ways to traverse the same path and only one of them goes through the edge $v_1 - v_2$. This will give us a edge betweenness centrality of $\frac{1}{2}$. Now there is a similar path from $v_{\frac{n}{2}+1}$ to v_2:

$$v_{\frac{n}{2}+1} - v_1 - v_2$$

$$v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2$$

Similarly as before, there are two ways to get between these two vertices with one way going through the colored edge. It will add a edge betweenness centrality of $\frac{1}{2}$. These will be the only paths on two edges that go through the colored edge because if we were to try to go to any other vertex while traversing a diametrical chord, we would be using more than 2 edges. Hence, the edge betweenness centrality from a path on 2 edges becomes $B_2'(e) = 2\left(\frac{1}{2}\right) = 1$.

Now consider the edge betweenness centrality on paths between two vertices on three edges which use the diametrical chord. Consider the path from v_1 to $v_{\frac{n}{2}+3}$. The path can be traversed in three ways:

$$v_1 - v_2 - v_3 - v_{\frac{n}{2}+3}$$

$$v_1 - v_2 - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3}$$

$$v_1 - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3}$$

We can see here that two of the three different paths have the colored edge $v_1 - v_2$. This will then give a edge betweenness centrality of $\frac{2}{3}$. Consider the path from v_n to $v_{\frac{n}{2}+2}$:

$$v_n - v_1 - v_2 - v_{\frac{n}{2}+2}$$

$$v_n - v_1 - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+2}$$

$$v_n - v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2}$$

Here we can see that the colored edge $v_1 - v_2$ only appears once so the edge betweenness centrality from this set of paths will be $\frac{1}{3}$. Consider the path from $v_{\frac{n}{2}+1}$ to v_3:

$$v_{\frac{n}{2}+1} - v_1 - v_2 - v_3$$

$$v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2 - v_3$$

$$v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3} - v_3$$

Similarly as the second set of points shown above, we can see that the colored edge $v_1 - v_2$ only appears once so the edge betweenness centrality from this set of paths will be $\frac{1}{3}$. Now look at the path from $v_{\frac{n}{2}}$ to v_2:

$$v_{\frac{n}{2}} - v_n - v_1 - v_2$$

$$v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_1 - v_2$$

$$v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2$$

Similarly as the first set of points from above, two of the three different paths have the colored edge $v_1 - v_2$. This will then give a edge betweenness centrality of $\frac{2}{3}$. We can say that these are the only such paths because if we try to traverse to a different vertex across the circle from v_1 outside of the already tested vertices, it will require more edges and hence be part of a different set of vertices. Adding the centralities together we get $B_3'(e) = 2\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right) = 2$.

Now consider the edge betweenness centrality of paths between two vertices on four edges with the use of a diametrical chord. Consider the path from v_1 to $v_{\frac{n}{2}+4}$. The path can be traversed in ways:

$$v_1 - v_2 - v_3 - v_4 - v_{\frac{n}{2}+4}$$

$$v_1 - v_2 - v_3 - v_{\frac{n}{2}+3} - v_{\frac{n}{2}+4}$$

$$v_1 - v_2 - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3} - v_{\frac{n}{2}+4}$$

$$v_1 - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3} - v_{\frac{n}{2}+4}$$

We can see that the colored edge $v_1 - v_2$ appears in 3 of the different paths. This means the edge betweenness centrality will be $\frac{3}{4}$. Now look at the paths from v_n to $v_{\frac{n}{2}+3}$:

$$v_n - v_1 - v_2 - v_3 - v_{\frac{n}{2}+3}$$

$$v_n - v_1 - v_2 - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3}$$

$$v_n - v_1 - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3}$$

$$v_n - v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3}$$

Here we can see that only two of the paths have the colored edge $v_1 - v_2$. This will have a edge betweenness centrality of $\frac{2}{4}$. Consider the path from v_{n-1} to $v_{\frac{n}{2}+2}$:

$$v_{n-1} - v_n - v_1 - v_2 - v_{\frac{n}{2}+2}$$

$$v_{n-1} - v_n - v_1 - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2}$$

$$v_{n-1} - v_n - v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2}$$

$$v_{n-1} - v_{\frac{n}{2}-1} - v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2}$$

Here we can see that only two of the paths have the colored edge $v_1 - v_2$. This will have a edge betweenness centrality of $\frac{1}{4}$. Consider the path from $v_{\frac{n}{2}+1}$ to v_4:

$$v_{\frac{n}{2}+1} - v_1 - v_2 - v_3 - v_4$$

$$v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2 - v_3 - v_4$$

$$v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3} - v_3 - v_4$$

$$v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2 - v_{\frac{n}{2}+3} - v_{\frac{n}{2}+4}$$

Like the third set of points we can see that only two of the paths have the colored edge $v_1 - v_2$. This will have a edge betweenness centrality of $\frac{1}{4}$. Consider the path from $v_{\frac{n}{2}}$ to v_3:

$$v_{\frac{n}{2}} - v_n - v_1 - v_2 - v_3$$

$$v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_1 - v_2 - v_3$$

$$v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2 - v_3$$

$$v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_{\frac{n}{2}+3} - v_3$$

Like the second set of points we can see that only two of the paths have the colored edge $v_1 - v_2$. This will have a edge betweenness centrality of $\frac{2}{4}$. Consider the path from $v_{\frac{n}{2}-1}$ to v_2:

$$v_{\frac{n}{2}-1} - v_{n-1} - v_n - v_1 - v_2$$

$$v_{\frac{n}{2}-1} - v_{\frac{n}{2}} - v_n - v_1 - v_2$$

$$v_{\frac{n}{2}-1} - v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_1 - v_2$$

$$v_{\frac{n}{2}-1} - v_{\frac{n}{2}} - v_{\frac{n}{2}+1} - v_{\frac{n}{2}+2} - v_2$$

Like the first set of points we can see that only two of the paths have the colored edge $v_1 - v_2$. This will have a edge betweenness centrality of $\frac{3}{4}$. We can say that these are the only such paths because if we try to traverse to a different vertex across the circle from v_1 outside of the already tested vertices, it will require more edges and hence be part of a different set of vertices. Adding the centralities together we get $B'_4(e) = 2\left(\frac{3}{4}\right) + 2\left(\frac{2}{4}\right) + 2\left(\frac{1}{4}\right) = 3$. So when we look at the centralities individually, no paths will have $bc = 1$. Since there are $k-1$ different spots where the path can be located, the betweenness centralities will decrease from $\frac{k-1}{k}, \frac{k-2}{k}, \cdots, \frac{1}{k}$. When all $2(k-1)$ centralities are added up, we end up getting

$$B'_k(e) = 2\left(\frac{k-1}{k}\right) + 2\left(\frac{k-2}{k}\right) + \cdots + 2\left(\frac{1}{k}\right) = \frac{k(k-1)}{k} = k-1$$

When we add all of the different betweenness centralities for the different edge lengths we will get

$$B'(e) = 2(B'_1(e) + B'_2(e) + B'_3(e) + \cdots + B'_k(e))$$

$$B'(e) = 2\left(\binom{k+1}{2} + 1 + 2 + 3 + ... + k - 1\right) = 2\left(\binom{k+1}{2} + \sum_{i=1}^{k-1} i\right)$$

$$B'(e) = 2\left(\frac{(k+1)!}{2!((k+1)-2)!} + \frac{(k-1)((k-1)+1)}{2}\right) = 2\left(\frac{(k+1)!}{2!(k-1)!} + \frac{(k-1)(k)}{2}\right)$$

$$B'(e) = 2\left(\frac{k(k+1)}{2} + \frac{(k-1)(k)}{2}\right) = 2\left(\frac{k^2+k}{2} + \frac{k^2-k}{2}\right)$$

$$B'(e) = 2\left(\frac{k^2+k+k^2-k}{2}\right) = 2\left(\frac{2k^2}{2}\right) = 2k^2$$

Thus it will hold for all k. ■

Now we will be looking at a slightly different type of circulant graph, still with two different chords. We will be looking at the combination of two circulant graphs. Namely C_{2n} and C_n. We have found that there are two distinct centralities on such graphs.

Theorem 6: *Consider the combination of two graphs C_{2n} and C_n such that every other vertex on the outer graph*

C_{2n} is connected with the inner graph C_n. The edge betweenness centrality of such a graph will be $B'(G) = n^2$. They will also always be bi-uniform in edge centrality.

Proof. We will proceed with proof by induction. Consider the case of $C_6 \cup C_3$ as follows:

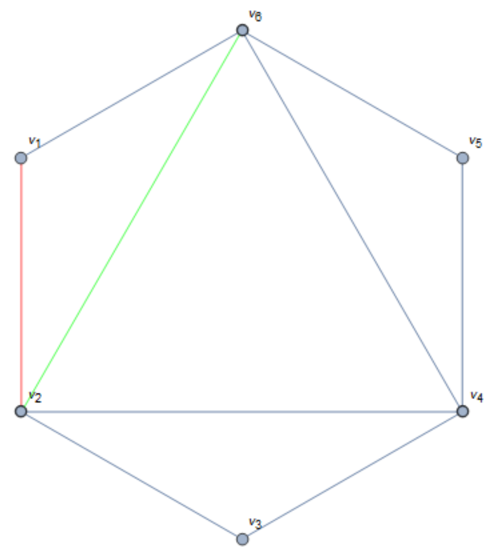


Figure 3: $C_6 \cup C_3$

Consider the edge $v_1 - v_2$. To find the edge betweenness centrality of this edge, look at all of the shortest paths which go through this edge. We can see that $v_1 - v_2$ and $v_1 - v_3$ give us an edge centrality value of 1 each towards the whole and $v_1 - v_4$ gives us a value of $\frac{1}{2}$. Total, the edge centrality would be $(2.5) * 2 = 5$. Now consider the edge $v_2 - v_6$.We have that the path $v_2 - v_6$ gives us an edge centrality of 1 towards the whole while $v_2 - v_5$ and $v_6 - v_3$ gives us a value of $\frac{1}{2}$. Total the centrality for this edge would be $(2) * 2 = 4$. This makes the total edge betweenness centrality for the graph to be $5 + 4 = 9 = 3^2$.

Now consider a graph of $C_{2n} \cup C_n$. It would be the vertices $v_1, v_2, v_3, ..., v_{2n}$ connected in a cycle with edges between $v_2 - v_4$, $v_4 - v_6$, ..., $v_k - v_{k+2}$, ..., $v_{2n-2} - v_{2n}$. We would have the edge $v_1 - v_2$, $v_{2n} - v_1 - v_2$, $v_{2n-1} - v2n - v_1 - v_2$, $\cdots$,vn+3 - v$_{n+4} - \cdots - v_1 - v_2$. When the beginning vertex 'backs-up' from beginning with v_1, we can say that the last backup will start with v_{n+3}. If we look at all of the vertices, we can see that since v_{n+1} is directly across the graph from v_1, we will have v_{n+2} be directly across the graph from v_2. Since the graph has symmetry, the path $v_2 - v_{n+2}$ can be traversed either way around the perimeter of the graph. Hence, we see that going from $v_2 - v_{n+3}$ would only have one shortest path meaning that it would be the last backup. The edge betweenness centrality will also include the half from the path $v_2 - v_{n+2}$. To get the total B' of this edge, we have $2n - (n+3) + 1$ edges which give us a centrality of 1. Then we have the single path of $v_2 - v_{n+2}$ giving us an edge centrality of $\frac{1}{2}$. This will be the only such edge because if we try to go to any other vertex v_i where $3 \geq i \geq n+2$ we will be able to skip over the vertex v_2 using the edge $v_1 - v_3$. Thus we see the edge betweenness centrality of the outer edge being

$$B_1'(e) = 2\left(2n - (n+3) + 1 + \frac{1}{2} + 1\right) = 2\left((n-2)\frac{1}{2} + 1\right) = 2n - 1$$

Now consider the case of the inner edge. By the makeup of the graph, we see that we have an edge which connects each of the vertices in a circular pattern. Then we have an edge that connects every other vertex starting from $v_1 - v_3$, $v_3 - v_5$, and so on. Because of this, we will see a pattern as to where the edge $v_1 - v_3$ will appear. We will look at the pattern by looking at whether we traverse a long edge (denoted 2) or a short edge (denoted 1). When we see the edge $v_1 - v_3$ we will denote it **2**. We are now looking at every edge that will give us an edge betweenness centrality of 1 which means the edge $v_1 - v_3$ will always be present. We would be able to find paths which look like:

$$\mathbf{2}, 1\mathbf{2}, \mathbf{2}1, \mathbf{2}2, 2\mathbf{2}, 1\mathbf{2}1, 1\mathbf{2}2, 12\mathbf{2}, \cdots$$

For the first case, we would just have **2**. For the second case we would have **2**, 1**2**, and **2**1. For the third case we would have **2**, 1**2**, **2**1, **2**2, 2**2**, and 1**2**1. We can see here that there is a pattern for every other case.

Consider the case where n is odd. We assume that this will work for $n = n-1$ and the edge betweenness centrality is $B' = \binom{n-2}{2}$. We will have the addition of $22\cdots2$ where the number of 2's is $\frac{n-1}{2}$. The value **2** could be in any of those spots so this would add $\frac{n-1}{2}$ to the edge centrality. We would also have $122\cdots21$ where the number of 2's is $\frac{n-3}{2}$. The value **2** could be in any of those spots so this would add $\frac{n-3}{2}$ to the edge centrality. In the case of $n-1$, we know the edge betweenness centrality is $B' = \binom{n-2}{2}$. So based off pattern, we know edge betweenness centrality for n is $B' = \binom{n-1}{2}$ and we will show why. Since all future cases encompass past cases, we can say:

$$B_n'(e) = B_{n-1}'(e) + ?$$

$$\binom{n-1}{2} = \binom{n-2}{2} + ?$$

$$\binom{n-1}{2} = \binom{n-2}{2} + \frac{n-1}{2} + \frac{n-3}{2}$$

$$n^2 + 3n + 6 + n - 1 + n - 3 = n^2 - 3n + 2$$

Multiplying all of the value out we do get this equation to be true.

Consider the case where n is even. Similarly as with the odd case, we know $B'_n(e) = B'_{n-1}(e)+?$. However, here we will have different additions to the edge centrality. We will have the addition of $122\cdots2$ and $22\cdots21$ where the number of 2's in both is $\frac{n-2}{2}$. So if we input this in the previous equation we get

$$\binom{n-1}{2} = \binom{n-2}{2}+?$$
$$\binom{n-1}{2} = \binom{n-2}{2} + \frac{n-2}{2} + \frac{n-2}{2}$$

since these fractions are equivalent to the previous ones, we will see that the equation will be true, hence making the statement true that the paths with $B' = 1$ are triangular numbers. Now consider the paths which add an edge centrality of $B' = \frac{1}{2}$. We know that we will get a half when the two end vertices of the path are directly across from each other in the graph. This will happen first with $v_1 - v_{n+1}$. Then we have the paths:

$$v_{2n} - v_n, v_{2n-1} - v_{n-1}, \cdots, v_{n+3} - v_3$$

We have $v_{n+3} - v_3$ as the last path because going further would not contain the edge $v_1 - v_3$ and hence, we would not need to count it. So to find the total centrality these halves would give us we do:

$$B' = \frac{1}{2}(n+1) - (3) + 1 = \frac{n-1}{2}$$

We add in a 1 at the end to make up for taking out an extra path when subtracting. So summing the total edge centrality of the inner edge we get:

$$B'_2(e) = 2\left(\binom{n-2}{2} + \frac{n-1}{2}\right) = 2\left(\frac{(n-2)(n-1)}{2} + \frac{n-1}{2}\right) = (n^2 - 3n + 2) + (n-1) = n^2 - 2n + 1$$

Summing both edge centralities we will find:

$$B'(G) = (n^2 - 2n + 1) + (2n - 1) = n^2$$

Hence, the equation for the centrality of the outside edge will hold for all values of n. ■

Proposition 7: *On the graph of $C_{2n} \cup C_n$, there will always be two distinct centralities in the graph.*

Each edge is connected to two vertices. There are two different kinds of vertices in this graph. One type of vertex, call it v_a, connects only two edges to one another. An example in the previous figure is v_1. The other type of vertex, call it v_b, connects 4 different edges to one another. An example in the previous figure is v_2. So in such a graph there will only ever be these two types of vertices because every other vertex is connected by the inner cycle. From this, we only have two different types of edges. Those connected with two vertices such as v_b or those connected with both vertices v_a and v_b. That would make this type of graph *bi-uniform* meaning it will always have a total of two distinct centralities. The uniformness of a graph is correlated to the number of different types of vertices and hence the number of different types of edges. ■

We just looked at circulant graphs with two different chords. Now consider circulant graphs with three different chords. We will still be focusing on graphs with $4k$ vertices. In the case of $C_{4k}(1,k,2k)$ we cannot look at all k values at once because the same rules do not apply throughout. So we will look at odd and even k separately. We will first consider the case of an even k.

Corollary 8: *Consider a circulant graph $C_{4k}(1,k,2k)$ with $k \in \mathbb{Z}$ and of the form $2m$ with $m \in \mathbb{Z}$. The edge betweenness centrality of the graph will be*

$$C_{4k} = k^2 + 4k - 1$$

Proof. We will prove each of the three different edges has a different edge betweenness centrality and we will show how to get each of them. Consider the following figure:

To find the total edge centrality we will consider three different sets of edges. We will look at $v_1 - v_2$ for the outer edge, $v_1 - v_5$ for the middle edge, and $v_1 - v_9$ for the diametrical edge. We will first consider the edge from v_1 to v_2. There are a few paths which add a value of 1 to the total edge centrality. Those paths are $v - 1 - v_2$, $v_1 - v_3$, and $v_{16} - v_2$. This adds 3 towards the edge centrality. Now consider paths on two edges. Those paths are $v_1 - v_6$, $v_1 - v_{14}$, $v_1 - v_{10}$, $v_5 - v_2$, $v_9 - v_2$, $v_{13} - v_2$. Each of these paths will add a value of $\frac{1}{2}$ to the edge centrality. This will add a total of 3 to the edge centrality. Now consider paths on three edges. Those paths will be $v_1 - v_7$, $v_{16} - v_6$, $v_1 - v_{11}$, $v_{16} - v_{10}$, $v_9 - v_3$, $v_8 - v_2$, $v_{13} - v_3$, and $v_{12} - v_2$. While each of these paths does have only 3 edges, there are six different ways to traverse between the two vertices. This happens because it can

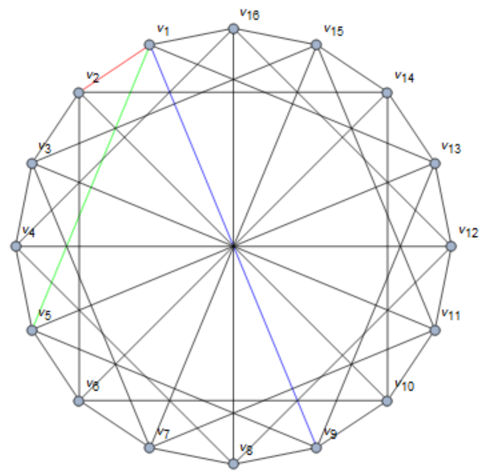


Figure 4: $C_{16}(1,4,8)$

be traversed through the middle edge as well as the diametrical edge. So each set of vertices either provides $\frac{2}{6}$ or $\frac{1}{6}$. All of these paths will give a total of $\frac{12}{6}=2$. In total, the edge betweenness centrality of the edge $v_1 - v_2$ will be $3+3+2=8$. When looking at how the centrality of this outer edge is derived, there is a pattern that can be seen in the table:

	1 path	2 path	3 path	4 path	5 path
C_{16}	3	3	$\frac{12}{6}$		
C_{24}	6	3	6	$\frac{24}{8}$	
C_{32}	10	3	6	9	$\frac{40}{10}$

We can see that the 1-path values increase by adding 2, then 3, and so on. Then we can see that the right off diagonal is multiples of 3. Finally, we see that the last term of each edge centrality can be derived from the value of m and turns out to simplify to equal m. Looking at the patterns together, we can see that the centrality for the outer edge will be

$$B_1'(e) = \frac{(2m)^2}{2} = \frac{k^2}{2} \tag{1}$$

So if we go back to C_{16} and input the values into the derived equation we get

$$\frac{4^2}{2} = 8$$

Hence we see that the formula holds in this case.

Now consider the circulant graph $C_{4k}(1, k, 2k)$. There will be several paths that can only be traversed in one way. They are as follows:

$$v_1 - v_2$$

$$v_1 - v_3$$

$$v_{4k} - v_2$$

$$\vdots$$

$$v_{4k-(m-2)} - v_2$$

These paths follow the circumference of the graph to it will take all such paths with 1 edge all the way up to m edges. This will give us $\sum_{i=1}^{m} i$ towards the edge centrality. Using the pattern of going through the desired edge and moving across the first middle edge for some paths, the diametrical edge for more paths, and the second middle edge for the final paths will get all of the other edges. The centralities for those edges will be as follows:

$$(3 + 6 + 9 + \cdots + 3(m-1)) + \frac{2m(m+1)}{2(m+1)}$$

The patterns for these values can be seen from the above table. When we combine the sums of the edge centralities together we get the following:

$$B_1'(e) = \sum_{i=1}^{m} i + 3 + 6 + 9 + \cdots + 3(m-1) + \frac{2m(m+1)}{2(m+1)}$$

$$B_1'(e) = \sum_{i=1}^{m} i + \sum_{i=1}^{m-1} 3i + \frac{2m(m+1)}{2(m+1)}$$

$$B_1'(e) = \frac{m(m+1)}{2} + 3\frac{(m-1)(m-1+1)}{2} + m$$

$$B_1'(e) = \frac{m^2 + m + 3m^2 - 3m + 2m}{2} = \frac{4m^2}{2} = \frac{(2m)^2}{2} = \frac{k^2}{2}$$

As we can see, our results match equation 1 above. We see that this technique will work for C_{4k}. So we can see here that equation 1 holds for all even values of k.

Consider $C_{16}(1,4,8)$ again. Now consider the edge $v_1 - v_5$ which we will call the middle edge. There will be one edge which will give the value of 1 towards the centrality of the middle edge and that edge is $v_1 - v_5$. There will be no such other edges for any number of vertices because going to any other vertices near the two chosen vertices can be done in multiple ways using different middle edges to 'cross over' that part of the graph. Next we have paths 2 edges. These would be $v_1 - v_6$, $v_1 - v_4$, $v_5 - v_2$, and $v_5 - v_{16}$. These paths will provide $\frac{1}{2}$ each towards the edge centrality with a total of 2 altogether. Next consider the paths with three edges. Those paths will be $v_1 - v_7$, $v_{16} - v_6$, and $v_{15} - v_5$. While each of these paths does have only 3 edges, there are six different ways to traverse between the two vertices. This happens because it can be traversed through the middle edge as well as the diametrical edge. Each set of vertices provides $\frac{1}{6}$. All of these paths will give a total of $\frac{3}{6} = \frac{1}{2}$. In total, the edge betweenness centrality of the edge $v_1 - v_5$ will be $1 + 2 + \frac{1}{2} = 3.5$. When looking at how the centrality of this middle edge is derived, there is a pattern that can be seen in the table:

	1 path	2 path	3 path	4 path	5 path
C_8	1	$\frac{2}{4}$			
C_{16}	1	2	$\frac{3}{6}$		
C_{24}	1	2	2	$\frac{4}{8}$	
C_{32}	1	2	2	2	$\frac{5}{10}$

We can see that there is always only 1 path which gives a value of 1. Then, after the first iteration, we see that the second path, which previously gave a value of $\frac{1}{2}$ gives a value of 2 and the fraction moves over to be $\frac{m+1}{2(m+1)}$. Using this information, we can see that the value of the edge centrality should be

$$B_2'(e) = 1 + 2(m-1) + \frac{(m+1)}{2(m+1)} = \frac{2k-1}{2} \tag{2}$$

So if we look back to C_{16} we can check the value of the edge centrality to be

$$1 + 2(2-1) + \frac{(2+1)}{2(2+1)} = 3 + \frac{3}{6} = 3.5$$

So we can see that the formula holds in this case. Now consider the circulant graph $C_{4k}(1,k,2k)$. There is only 1 path which can only be traversed in one way. It is $v_1 - v_{k+1}$. For the paths that can be traversed in two different ways, they will look like the following:

$$v_1 - v_{k+2}$$

$$v_1 - v_k$$

$$v_{k+1} - v_2$$

$$v_{k+1} - v_{4k}$$

This pattern will stay throughout and there will be no other paths which give $\frac{1}{2}$ to the edge centrality because there are no other ways to have 3 vertices and 2 edges through the specific middle path. We will see a similar trend with 3 edges each giving $\frac{1}{3}$, all the way until m edges. When we get to $m+1$ edges, we don't get 2 from it but $\frac{(m+1)}{2(m+1)}$ because half of the paths can be traversed like above but the other half can each be traversed on the outer edge. So if we calculate the total edge centrality we get

$$B_2'(e) = 1 + (2 + 2 + ... + 2) + \frac{(m+1)}{2(m+1)}$$

$$B_2'(e) = 1 + 2(m-1) + \frac{(m+1)}{2(m+1)}$$

$$B_2'(e) = 1 + 2(m-1) + \frac{1}{2}$$

So we can see here that equation 2 holds for all even values of k.

Consider $C_{16}(1, 4, 8)$ again. Now consider the edge $v_1 - v_9$ which we will call the diametrical edge. There will be one edge which will give the value of 1 towards the centrality of the diametrical edge and that edge is $v_1 - v_9$. There will be no such other edges for any number of vertices because going to any other vertices near the two chosen vertices can be done in multiple ways using different diametrical edges to 'cross over' that part of the graph. Next we have paths 2 edges. These paths are $v_1 - v_8$, $v_1 - v_{10}$, $v_9 - v_2$, and $v_9 - v_{16}$. These paths will provide $\frac{1}{2}$ each towards the edge centrality with a total of 2 altogether. Next consider the paths with three edges. Those paths will be $v_1 - v_{11}$, $v_{16} - v_{10}$, $v_{15} - v_9$, $v_1 - v_7$, $v_2 - v_8$, and $v_3 - v_9$. While each of these paths does have only 3 edges, there are six different ways to traverse between the two vertices. This happens because it can be traversed through the middle edge as well as the diametrical edge. Each set of vertices provides $\frac{1}{6}$. All of these paths will give a total of $\frac{6}{6} = 1$. In total, the edge betweenness centrality of the edge $v_1 - v_9$ will be $1 + 2 + 1 = 4$. When looking at how the centrality of this diametrical edge is derived, there is a pattern that can be seen in the table:

	1 path	2 path	3 path	4 path	5 path
C_8	1	$\frac{4}{4}$			
C_{16}	1	2	$\frac{6}{6}$		
C_{24}	1	2	2	$\frac{8}{8}$	
C_{32}	1	2	2	2	$\frac{10}{10}$

We can see that there is always only 1 path which gives a value of 1. Then, after the first iteration, we see that the second path, which previously gave a value of 1 gives a value of 2 and the previous value moves over to be $\frac{2(m+1)}{2(m+1)}$ on $m+2$ edges. Using this information, we can see that the value of the edge centrality should be

$$B_3'(e) = 1 + 2(m-1) + \frac{2(m+1)}{2(m+1)} = \frac{2k}{2} \tag{3}$$

So if we look back to C_{16} we can check the value of the edge centrality to be

$$1 + 2(2-1) + \frac{2(2+1)}{2(2+1)} = 3 + \frac{6}{6} = 4$$

So we can see that the formula holds in this case. Now consider the circulant graph $C_{4k}(1, k, 2k)$. There is only 1 path which can only be traversed in one way. It is $v_1 - v_{2k+1}$. For the paths that can be traversed in two different ways, they will look like the following:

$$v_1 - v_{2k+2}$$

$$v_1 - v_{2k}$$

$$v_{2k+1} - v_2$$

$$v_{2k+1} - v_{4k}$$

This pattern will stay throughout and there will be no other paths which give $\frac{1}{2}$ to the edge centrality because there are no other ways to have 3 vertices and 2 edges through the specific middle path. We will see a similar trend with 3 edges each giving $\frac{1}{3}$, all the way until m edges. When we get to $m+1$ edges, we don't get 2 from it but $\frac{2(m+1)}{2(m+1)}$ because half of the paths can be traversed like above but the other half can each be traversed on the outer edge. So if we calculate the total edge centrality we get

$$B_3'(e) = 1 + (2 + 2 + ... + 2) + \frac{2(m+1)}{2(m+1)}$$

$$B_3'(e) = 1 + 2(m-1) + \frac{2(m+1)}{2(m+1)}$$

$$B_3'(e) = 1 + 2(m-1) + 1$$

So we can see here that equation 3 holds for all even values of k.

Now if we take the sum of all 3 centralities, we will be able to find the equation for finding the total edge betweenness centrality of the whole graph. Combine equations 1,2, and 3 as follows:

$$B'(G) = 2\left[\frac{k^2}{2} + \frac{2k-1}{2} + \frac{2k}{2}\right]$$

$$B'(G) = k^2 + 2k - 1 + 2k = k^2 + 4k - 1 \tag{4}$$

■

Now we will consider the case of the odd k. We will see that the equation for these centralities is slightly more involved because it has to do with m rather than k directly.

Corollary 9: *If you have a circulant graph $C_{4k}(1, k, 2k)$ with $k \in \mathbb{Z}$ and of the form $2m + 1$ with $m \in \mathbb{Z}$, the edge betweenness centrality of the graph will be*

$$C_{4k} = 2\left(\frac{2m^3 + 10m^2 + 14m + 4}{m+2}\right)$$

Proof. We will go about proving each of the three different edges has a different edge betweenness centrality and we will show how we get each of them. Consider the following circulant graph:

To find the total edge betweenness centrality we will find the edge betweenness centrality of the outer edge by looking at the shortest paths through $v_1 - v_2$, then we will look at the middle edge by finding the shortest paths through $v_1 - v_4$, and finally we will look at the diametrical edge $v_1 - v_7$ to find the number of shortest paths there. Consider the edge $v_1 - v_2$. There is only one path that can go through $v_1 - v_2$ with only one way to take that path. That path will be $v_1 - v_2$. This will add a value of 1 to the total centrality of the edge. Now

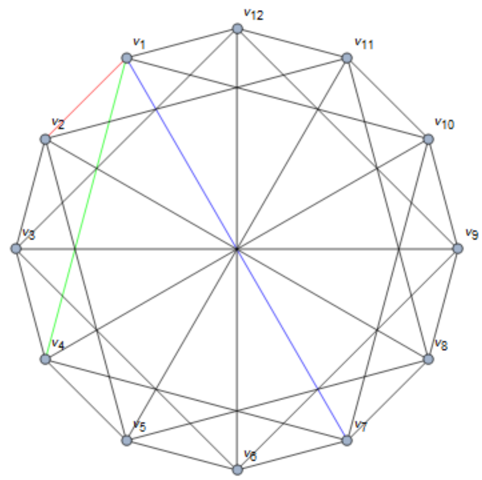


Figure 5: $C_{12}(1,3,6)$

consider paths that can each be traversed in two different ways. Those paths will be $v_1 - v_5$, $v_1 - v_8$, $v_7 - v_2$, and $v_{10} - v_2$. Since in each pair of paths, the edge $v_1 - v_2$ appears once. This means each path pair will give the value of $\frac{1}{2}$ to the total edge centrality. These 4 paths combined add 2 to the total value of the edge centrality. Finally, there are 4 different paths which can be traversed in 3 different ways. Those paths are $v_1 - v_{11}$, $v_1 - v_3$, $v_{12} - v_2$, and $v_4 - v_2$. For each set of 3 paths, the edge $v_1 - v_2$ appears only once which means each set provides $\frac{1}{3}$ each. So these edges provide $1\frac{1}{3}$ to the total edge centrality. In total, the edge betweenness centrality of the edge $v_1 - v_2$ will be $B_1'(e) = 2(1 + 2 + 1\frac{1}{3}) = 2(4\frac{1}{3}) = 8\frac{2}{3}$. When looking at how the centrality of this outer edge is derived, there is a pattern that can be seen in the table:

	1 path	2 path	3 path	4 path	5 path	6 path
C_{12}	1	2	$1\frac{1}{3}$			
C_{20}	3	3	4	$2\frac{1}{4}$		
C_{28}	6	3	6	6	$3\frac{1}{5}$	
C_{36}	10	3	6	9	8	$4\frac{1}{6}$

We can see that the 1-path values increase by adding 2, then 3, and so on. Then we can see that the main diagonal is multiples of 3 and that pattern will continue down the table. We can also see that the diagonal to the right of the main diagonal is multiples of 2 which will continue down the table. The last addition to the edge centrality which contains the fraction also has a pattern. The whole number increases by 1 every time and the denominator also increases by 1 each time. Looking down the columns of the table, we can see that the number stops changing after the second value and that is because there are no other such paths no matter how many vertices are added to the graph. Looking at the patterns together, we can see that the centrality for the outer edge will be

$$B_1'(e) = 4\sum_{i=1}^{m} i + \frac{1}{m+2} = \frac{4m(m+1)}{2} + \frac{1}{m+2} = 2m(m+1) + \frac{1}{m+2} \tag{5}$$

So if we go back to C_{12} and input the values into the derived equation we get

$$2\left(2(1)(1+1) + \frac{1}{1+2}\right) = 2\left(4\frac{1}{3}\right) = 8\frac{2}{3}$$

So we can see that the formula holds in this case. Now consider the circulant graph $C_{4k}(1, k, 2k)$. There will be several paths that can only be traversed in one way. They are as follows:

$$v_1 - v_2$$

$$v_1 - v_3$$

$$v_{4k} - v_2$$

$$\vdots$$

$$v_{4k-(m-2)} - v_2$$

These paths follow the circumference of the graph to it will take all such paths with 1 edge all the way up to m edges. This will give us $\sum_{i=1}^{m} i$ towards the edge centrality. The value we get from the other sets of paths will be as follows:

$$(3 + 6 + 9 + \cdots + 3(m-1)) + (2m) + \left(m + \frac{1}{m+2}\right)$$

The patterns for these values can be seen from the above table. When we combine the sums of the edge centrality all together we get the following:

$$B_1'(e) = \sum_{i=1}^{m} i + 3 + 6 + 9 + \cdots + 3(m-1) + 2m + m + \frac{1}{m+2}$$

$$B_1'(e) = \sum_{i=1}^{m} i + \sum_{j=1}^{m-1} 3j + 2m + m + \frac{1}{m+2}$$

$$B_1'(e) = \frac{m(m+1)}{2} + 3\frac{(m-1)((m-1)+1)}{2} + 3m + \frac{1}{m+2}$$

$$B_1'(e) = \frac{m^2 + m + 3m^2 - 3m + 6m}{2} + \frac{1}{m+2}$$

$$B_1'(e) = \frac{4m^2 + 4m}{2} + \frac{1}{m+2}$$

$$B_1'(e) = 2m^2 + 2m + \frac{1}{m+2} = 2m(m+1) + \frac{1}{m+2}$$

As we can see, our results match equation 5 above. We see that this technique will work for C_{4k}. So we can see here that equation 5 holds for all odd values of k.

Consider again $C_{12}(1,3,6)$. We will consider the edge $v_1 - v_4$ which we will call a middle edge. There will be one edge which will give the value of 1 towards the centrality of the middle and and that edge is $v_1 - v_4$. There will be no such other edges for any number of vertices because going to any other vertices near the two chosen vertices can be done in multiple ways using different middle edges to 'cross over' that part of the graph. Next we have paths with 2 different ways to traverse them. These would be $v_1 - v_5$ and $v_{12} - v_4$. These are the only two such paths because if we try to go from v_2 and cross over the middle edge, there would be an extra third way to traverse that path by going along the outer edge. So the two above paths with provide $\frac{1}{2}$ each giving a total of 1 to the edge centrality. Now look at the special cases. We have the two paths $v_2 - v_4$ and $v_1 - v_3$. These paths can be traversed the same as $v_1 - v_5$ but they can also be traversed along the edge of the graph. This means each of these paths will provide $\frac{1}{3}$ towards the edge centrality with a total of $\frac{2}{3}$. Hence, we have the total edge betweenness centrality of the middle edge to be $2\left(1+1+\frac{2}{3}\right) = 5\frac{1}{3}$. When we look at how the centrality of this middle edge is derived, there is a pattern that can be seen in the table:

	1 path	2 path	3 path	4 path	5 path	6 path
C_{12}	1	1	$\frac{2}{3}$			
C_{20}	1	2	1	$\frac{3}{4}$		
C_{28}	1	2	2	1	$\frac{4}{5}$	
C_{36}	1	2	2	2	1	$\frac{5}{6}$

We can see that there is always only 1 path which gives a value of 1. Then, after the first iteration, we see that the second path, which previously gave a value of 1 gives a value of 2. We can also see that the paths which can be traversed along the outer edges give us values of $\frac{m+1}{m+2}$. Using this information, we can see that the value of the edge centrality should be

$$B_2'(e) = 2 + 2(m-1) + \frac{m+1}{m+2} \tag{6}$$

So if we look back to C_{12} we can check the value of the edge centrality to be

$$2\left[2 + 2(1-1) + \frac{1+1}{1+2}\right] = 2\left[2\frac{2}{3}\right] = 5\frac{1}{3}$$

So we can see that the formula holds in this case. Now consider the circulant graph $C_{4k}(1, k, 2k)$. There is only 1 path which can only be traversed in one way. It is $v_1 - v_{k+1}$. For the paths that can be traversed in two different ways, they will look like the following:

$$v_1 - v_{k+1} - v_k, v_1 - v_{4k} - v_k$$

$$v_1 - v_{k+1} - v_{k+2}, v_1 - v_2 - v_{k+2}$$

$$v_{4k} - v_1 - v_{k+1}, v_{4k} - v_k - v_{k+1}$$

$$v_2 - v_1 - v_{k+1}, v_2 - v_{k+2} - v_{k+1}$$

This pattern will stay throughout and there will be no other paths which give $\frac{1}{2}$ to the edge centrality because there are no other ways to have 3 vertices and 2 edges through the specific middle path. We will see a similar trend with 4 vertices and 3 edges each giving $\frac{1}{3}$, all the way until $m+1$ vertices and m edges. When we get to $m+2$ vertices and $m+1$ edges, we don't get 2 from it but 1 because half of the paths can be traversed like above but the other half can each be traversed on the outer edge. The reason for the consistent fraction at the end is because with $m+1$ edges, there can only be $m+1$ different paths. So when there are $m+2$ different paths on $m+1$ edges, there will only be $m+1$ different paths giving the value $\frac{m+1}{m+2}$ to the edge centrality. So if we calculate the total edge centrality we get

$$B_2'(e) = 2 + (2 + 2 + ... + 2) + \frac{m+1}{m+2} = 2 + \sum_{i=1}^{m-1} 2 + \frac{m+1}{m+2} = 2 + 2(m-1) + \frac{m+1}{m+2}$$

So we can see here that equation 6 holds for all odd values of k.

Consider $C_{12}(1,3,6)$. We will consider the edge $v_1 - v_7$ which we will call a diametrical edge. There will only be one edge which can be traversed in one way and that is $v_1 - v_7$. There will be no such other edges for any number of vertices because going to any other vertices near the two chosen vertices can be done in multiple ways using different diametrical edges to 'cross over' that part of the graph. Next we have paths with 2 different ways to traverse them. These would be $v_1 - v_8$, $v_1 - v_6$, $v_2 - v_7$, and $v_{12} - v_7$. These are the only such paths because if we try to add an edge, there would be more than two ways to traverse that path. So we get a value of $\frac{1}{2}$ from each of these paths totaling in a value of 2 towards the edge centrality. If we try to traverse something with 3 edges through the diameter, we will find that it can be traversed by other edges much easier. So if we total the edge betweenness centrality, we will get $2(1+2) = 6$. When we look at how the centrality of this diametrical edge is derived, there is a pattern that can be seen in the table:

	1 path	2 path	3 path	4 path	5 path	6 path
C_{12}	1	2				
C_{20}	1	2	2			
C_{28}	1	2	2	2		
C_{36}	1	2	2	2	2	

We can see that there will always be only one edge which provides a value of 1. We can also see that every other set of paths will provide 2 to the edge centrality. Using this information, we can see that the value of the edge centrality should be

$$B'_3(e) = 1 + 2m \tag{7}$$

So if we look back to C_{12} we can check the value of the edge centrality to be

$$2[1 + 2(1)] = 2[3] = 6$$

So we can see that the formula holds in this case. Now consider the circulant graph $C_{4k}(1,k,2k)$. There is only 1 path which can only be traversed in one way. It is $v_1 - v_{2k+1}$. For the paths that can be traversed in two different ways, they will look like the following:

$$v_1 - v_{2k+1} - v_{2k}, v_1 - v_{4k} - v_{2k}$$

$$v_1 - v_{2k+1} - v_{2k+2}, v_1 - v_2 - v_{2k+2}$$

$$v_{4k} - v_1 - v_{2k+1}, v_{4k} - v_{2k} - v_{2k+1}$$

$$v_2 - v_1 - v_{2k+1}, v_2 - v_{2k+2} - v_{2k+1}$$

We can see that this is similar to how the middle edge behaves. This pattern will stay throughout and there will be no other paths which give $\frac{1}{2}$ to the edge centrality because there are no other ways to have 3 vertices and 2 edges through the specific middle path. We will see a similar trend with 4 vertices and 3 edges each giving $\frac{1}{3}$, all the way until $m+2$ vertices and $m+1$ edges. So if we calculate the total edge centrality we get

$$B_3'(e) = 1 + (2 + 2 + ... + 2) = 1 + \sum_{i=1}^{m} 2 = 1 + 2m$$

So we can see here that equation 7 holds for all odd values of k.

If we take the sum of all 3 centralities, we will be able to find the equation for finding the total edge betweenness centrality of the whole graph. Combine equations 5,6, and 7 as follows:

$$B'(G) = 2\left[\left(2m(m+1) + \frac{1}{m+2}\right) + \left(2 + 2(m-1) + \frac{m+1}{m+2}\right) + (1+2m)\right]$$

$$B'(G) = 2\left[\frac{2m(m+1)(m+2) + 1 + 2(m+2) + 2(m-1)(m+1) + (m+1) + (m+2) + 2m(m+2)}{m+2}\right]$$

$$B'(G) = 2\left[\frac{2m^3 + 6m^2 + 4m + 1 + 2m + 4 + 2m^2 + 2m - 4 + m + 1 + m + 2 + 2m^2 + 4m}{m+2}\right]$$

$$B'(G) = 2\left[\frac{2m^3 + 10m^2 + 14m + 4}{m+2}\right] \tag{8}$$

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5 Future Work

There are many different directions that our work can go from here. Apart from circulant graphs, there are many other families of graphs that can be explored. Also within the family of circulant graphs there are many sub-families. Some extensions of our work include but are not limited to:

- Patterns concerning $C_{nk} \cup C_k$ where n is a value greater than 2.
- Patterns concerning circulant graphs where all centralities are different.
- Patterns in circulant graphs with more than three chords.
- Patterns between betweenness centrality and edge betweenness centrality.

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