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Optimal L(h, k) labelings of Cartesian products of complete graphs and paths

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by

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Abstract

In an $L(h,k)$ labeling of a graph G we assign non-negative integers to the vertices of the graph such that the labels of the vertices that are at a distance of one have a difference of at least h and the labels of the vertices which are at a distance of two have a difference of at least k. The aim in general is to minimize the $L(h,k)$ span, where the $L(h,k)$ span is the difference between highest and lowest label used. In this thesis we analyze $L(h,k)$ labelings of Cartesian products of complete graphs and path. For $h \geq k$ we establish the minimum $L(h,k)$ span of these graphs. For $h < k$ we establish the minimum $L(h,k)$ span in most cases. We provide conjectures on the minimum $L(h,k)$ spans for values of h and k for which we were not able to establish the minimum $L(h,k)$ span.

Contents

1. Introduction and definitions

The $L(h,k)$ problem has its origins in the frequency assignment problem. It is typically a problem of optimization. Let us say that there are some radio transmitters for which we need to assign frequencies with an aim of minimizing the highest frequency used. The requirements originate from the fact that frequencies of transmitters which are close by can possibly interfere with each other. Therefore the prime objective of this frequency assignment problem was to eliminate interference. During the 1960's the demand for usable spectrum increased exponentially. Simultaneously the growth of the usable spectrum decreased. This led to development of a number of new models for frequency assignment with an objective of minimizing the frequencies used and eliminating interferences, if any. One such model was $L(h,k)$ labeling [20].

There are two types of objective functions in frequency assignment problems. One is called the minimum span problem (where span is the difference between highest frequency and lowest frequency) [15]. The other is called the minimum order problem (where order is the number of distinct frequencies used). Minimizing the order does not always minimize the highest frequency used. While we can find the order using a computational method in polynomial time, it is not always possible to find the minimum span in polynomial time. In both types of problems we have frequency-distance constraints. That is, transmitters that are close by should not have interfering frequencies [13].

Metzger [18] first established the fact that the frequency assignment problem can be solved using graph theory. He developed two coloring procedures for the problem and talked about two techniques to solve the problem. One technique is called the frequency exhaustive technique and the other is called the uniform assignment technique. Thus a model was created which considerably increased the spectrum savings.

Hale [13] also pointed out the connection between the frequency assignment problem and graph theory and used well-known graph theory algorithms to solve the frequency assignment problem. Later, a formal definition using graph theory terminologies was developed to analyze the frequency assignment problem.

1.1 Overview

Section 1.3 contains basic definitions and notations used in the thesis. In the section we also define Cartesian product of the graphs with examples. Also we introduced a new term called transition.

In Section 2, we give a brief introduction to $L(h,k)$ coloring and discuss the NP completeness of the problem. We also provide some known results pertaining to $L(h,k)$ labeling.
Section 3 contains main results of this thesis. Section 3.1 starts with the basic lemma required for our results. We divide the problem into different cases depending on the values of h and k. As we increase the value of k keeping h as constant the problem becomes more and more tedious owing to fact that we need to analyze more layers of a complete graph.
Section 4 contains lower bounds for the span of other graphs. In Section 5 and 6 we discuss future directions in which we can proceed.

1.2 Definitions

A graph G is a set of vertices (nodes) V connected by edges (links) E. Thus $G = (V, E)$. The vertex set is denoted by $V(G)$ and the edge set is denoted by $E(G)$. An edge e is a link between two vertices. Two vertices are said to be *adjacent* if they have an edge connecting them. The *degree* of a vertex is the number of edges that are incident to the vertex and is denoted by $\deg(v)$. The maximum degree of a graph G, denoted by $\Delta(G)$, is the maximum across the vertices of G of the degrees of those vertices. A *leaf* of a graph is a vertex with degree one. Leaves may not always be present in a graph. A graph H is called a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $G = (V, E)$ be any graph, and let $S \subseteq V$ be any subset of vertices of G, then the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. The *distance* between two vertices u and v (represented by $d(u, v)$) is the number of edges in the shortest path connecting them.

A *complete graph* on n vertices is a graph in which every pair of distinct vertices is adjacent. The complete graph on n vertices is denoted by K_n. A graph is *connected* if there is a path between every pair of vertices. A path is a connected graph in which there are two leaves and all other vertices have degree two. A path on n vertices is denoted by P_n. To define the *Cartesian product*, consider two graphs G and H. Suppose the vertices of graph G are $v_1, v_2, \ldots, v_n$ and that of H are $u_1, u_2, \ldots, u_m$. The vertex set of the Cartesian product of the graph G and H is the set of vertices $V_G \times V_H$. The vertex $w_1 = (u_i, v_j)$ is adjacent to $w_2 = (u_k, v_l)$ if either u_i and u_k are same and v_j is adjacent to v_l, or v_j and v_l are same and u_i is adjacent to u_k. The Cartesian product of G and H is defined by $G \square H$. Consider the Cartesian product of K_n and P_m. By the above definition, the Cartesian product contains m layers of K_n. Each vertex v_i is adjacent to its corresponding image in the next layer. Throughout this paper we use the following representation. By $v_{i,j}$ we mean a vertex in the Cartesian product where j represents the layer to which the vertex belongs and i represents the position of the vertex in that layer. For example, consider Figure 1 which is $K_5 \square P_2$. We have two layers of K_5. As we can see in Figure 1 any vertex in the first layer will be named $v_{i,1}$ and any vertex in the second layer will be named $v_{i,2}$. Now the graph $K_5 \square P_2$ will have an edge between vertex $v_{1,1}$ in the first layer and the vertex $v_{1,2}$ in the second layer. Similarly there will be an edge between the vertex $v_{2,1}$ in the first layer and vertex $v_{2,2}$ in the second layer. In general for $K_n \square P_m$ (when $m > 1$), there will be an edge between the vertices $v_{i,j}$ and $v_{i,j+1}$ for $j < m$.

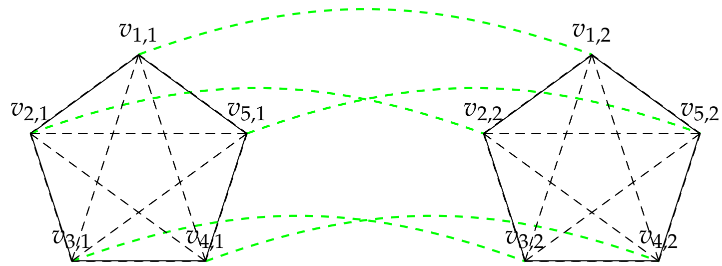

Figure 1: $K_5 \square P_2$

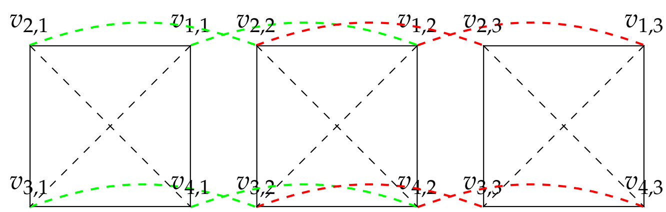

Figure 2: $K_4 \square P_3$

$K_4 \square P_3$ is given in Figure 2. Note that the vertex $v_{1,1}$ is at a distance of two from the vertex $v_{1,3}$. Throughout the thesis for Cartesian product of $K_n \square P_m$ we draw layers side by side and do not show the edges between adjacent layers. Also we do not show all the edges in K_n to avoid congestion. For example the graph $K_n \square P_3$ is represented as shown in Figure 3.

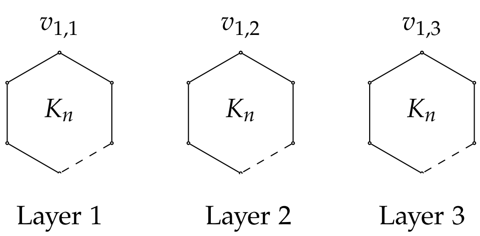

Figure 3: $K_n \square P_3$

Given two non negative integers h and k, an $L(h,k)$ *labeling* of a graph $G = (V, E)$ is a map from V to a set of labels such that adjacent vertices receive labels at least h apart, while vertices at distance two receive labels at least k apart. The $L(h,k)$ span of a labeling is difference between the highest label and the lowest label. In this thesis, we use 0 as the smallest label and hence the span of an $L(h,k)$ labeling is the highest label used.

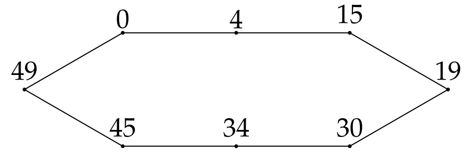

Figure 4: $L(4,15)$ labeling of C_8

An example of an $L(4,15)$ labeling is shown in Figure 4. The highest label is 49. Hence the span of this labeling is 49. The $L(h,k)$ span of a graph, denoted by, $\lambda_{h,k}(G)$ is minimum span of all $L(h,k)$ labelings for a graph. When no confusion will result, we refer to $\lambda_{h,k}(G)$ simply as $\lambda(G)$. Figure 5 shows the $L(h,k)$ span for C_8. So we have $\lambda_{4,15}(C_8) = 19$.

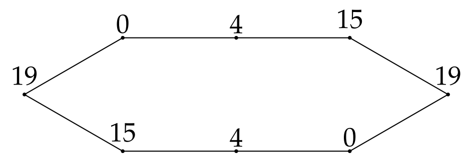

Figure 5: $L(4,15)$ labeling of C_8

We introduce a new term called *transition*. A transition is defined for two adjacent layers of $K_n \Box P_m$. Let f be an $L(h,k)$ labeling of $K_n \Box P_2$. Arrange the vertices in increasing order of their labels. Let the order be $v_1, v_2, \ldots, v_{2n-1}, v_{2n}$. That is $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{2n-1}) < f(v_{2n})$. The vertex v_i can belong to either the first layer or the second. We say we have a transition if the vertex v_i belongs to one layer and the vertex v_{i+1} to the other layer. Depending on the number of transitions it has an $L(h,k)$ labeling can be either a one-transition or a two-transition or in general an n-transition labeling. To illustrate the concept, consider an $L(4,15)$ labeling of $K_6 \Box P_2$ as shown in Figure 6.

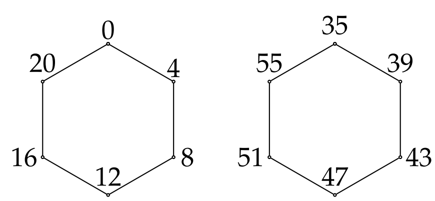

Figure 6: $L(4,15)$ labeling for $K_6 \Box P_2$

In this example when we arrange the vertices in the increasing order of their labels, the vertices of the first layer come first followed by the vertices of second layer. Therefore we have only one transition. The transition occurs since label 20 is in the first layer and the next highest label 35 is in the second layer.

Now consider Figure 7 which shows a two-transition $L(4,15)$ labeling of $K_6 \Box P_2$.

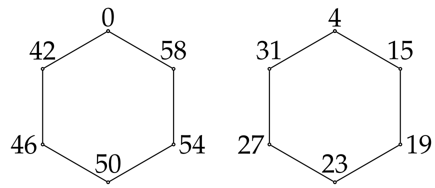

Figure 7: $L(4,15)$ labeling for $K_6 \Box P_2$

When we arrange the vertices in increasing order of their labels, we have 0 in the first layer, then we have 4 in the second layer. Therefore we have one transition there. After label 31 which belongs to the second layer we have label 42 in the first layer. Therefore we have another transition. Hence this labeling has two transitions.

Figure 8 shows a three- transition $L(4,15)$ labeling of $K_6 \Box P_2$:

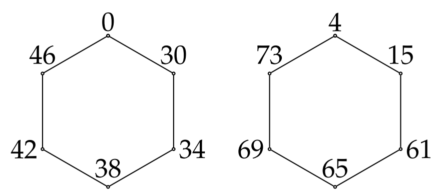

Figure 8: $L(4,15)$ labeling for $K_6 \Box P_2$

In this labeling we have 0 in the first layer and 4 in the second layer which is the first transition. After the label 15 which is in second layer the next highest label 30 in first layer. Hence we have the second transition. The label 46 is in first layer and 61 is in second layer which is the third transition.

Let f be an $L(h,k)$ labeling of $K_n \Box P_2$. Let $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let us say we have a transition at some vertex v_i. Without loss of generality assume v_i is in first layer and v_{i+1} is in second layer. Suppose that v_{i-1}(if it exists) is in first layer and that v_{i+2}(if it exists) is in second layer. Then it is called an *isolated transition*.

Suppose we have a set of vertices v_i, v_{i+1}, v_{i+2} such that v_i is in first layer, v_{i+1} is in second layer and v_{i+2} is in first layer. Suppose that v_{i-1} (if it exists) is in first layer and that v_{i+2} (if it exists) is in second layer. Then it is called *combined two transition system*.

In general if we have n consecutive transitions starting at v_{i_1} then we say we have *combined n transition system*.

Observation 1. *Consider an $L(h,k)$ labeling of a graph K_n. Since the smallest label in a layer cannot be less than zero and also we have n vertices which are adjacent to each other, the highest label in the layer is at least $(n-1)h$. In fact, $\lambda_{h,k}(K_n) = (n-1)h$.*

2. Background on $L(h,k)$ labeling of various graphs

2.1 Introduction

The $L(h,k)$ problem was first introduced by Griggs and Yeh [12]. The problem arises in multihop radio networks for frequency assignment. Initially the problem was studied for only special case where $h = 2$ and $k = 1$. Later on the $L(h,k)$ labeling was formally defined and was used to model many other problems. $L(h,k)$ labelings are used in packet radio networks to avoid hidden collisions. They are also used in optical cluster based networks [19].

There are many results pertaining to $L(h,k)$ span of different graphs. The upper and lower bounds were established in special cases where $(h,k) = (1,0),(2,1)$ for any generalized graph. It was proved that $\lambda_{0,1}(G) \leq \Delta^2 - \Delta$, where Δ is the maximum degree for any graph G [14].

In the case of $L(2,1)$ labeling of a graph we have the lower bound, $\lambda_{2,1}(G) \geq \Delta + 1$ [10]. Griggs and Yeh gave an upper bound, which is $\lambda_{2,1} \leq \Delta^2 + 2\Delta$ [12]. It is conjectured that $\lambda_{2,1}(G) \leq \Delta^2$ [12]. This still remains an open problem. For a complete list of results about $L(h,k)$ labeling and its history, see [3].

2.2 NP completeness of the problem

The $L(h,k)$ labeling problem in general is NP hard and becomes computationally intractable. It is conjectured that the problem of finding if the $L(h,k)$ span is less than some number r is NP complete. Bertossi and Bonuccelli showed that $L(0,1)$ labeling can be used to solve the problem of avoiding hidden collisions in packet radio networks [2]. They also proved that the decision version of $L(0,1)$ is NP complete. McCormick proved that the decision version of $L(1,1)$ labeling for minimum span is equivalent to decision version of $L(2,1)$ labeling for minimum order [17].

Griggs and Yeh proved that the decision version of the $L(2,1)$ labeling problem is NP complete [12].

2.3 Upper and lower bounds

For any positive integers h and k we have $\lambda_{h,k}(G) \geq h + (\Delta+1)k$, where Δ is maximum vertex degree [4].

When $k = 1$, $\lambda_{h,1}(G) \leq \Delta^2 + (h-1)\Delta$ [10]. Later it was proved that $\lambda_{h,1}(G) \leq \Delta^2 + (h-1)\Delta - 2$, when $\Delta \geq 3$ [10]. Also it was observed that $\lim_{h\to\infty} \frac{\lambda_{h,1}(G)}{\lambda_{h+1,1}(G)} = 1$ [4].

2.4 Known results

Theorem 2.1. [16] $\lambda_{0,1}(P_n) = 1$.

In Figure 9 an optimal $L(0,1)$ labeling of P_6 is shown.

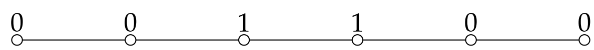

Figure 9: Optimal $L(0,1)$ labeling of P_6

Theorem 2.2. [1] $\lambda_{1,1}(P_2) = 1$ *and* $\lambda_{1,1}(P_n) = 2$ *for* $n > 2$.

Figure 10 shows an optimal $L(h,k)$ labeling.

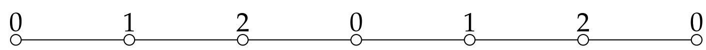

Figure 10: Optimal $L(1,1)$ labeling of P_7

Theorem 2.3. [11] *For any h and k we have.*

$$\lambda_{h,k}(P_n) = \begin{cases} 0, & \textit{for } n = 1 \\ h, & \textit{for } n = 2 \\ h+k, & \textit{for } 3 \leq n \leq 4 \\ h+2k, & \textit{for } n \geq 5 \textit{ and } h \geq 2k \\ 2h, & \textit{for } n \geq 5 \textit{ and } h < 2k \end{cases}$$

Theorem 2.4. [7] *In case of a cycle we have:*

$$\lambda_{0,1}(C_n) = \begin{cases} 1, & \textit{if } n \equiv 0 \pmod 4 \\ 2, & \textit{otherwise} \end{cases}$$

Figures 11 and 12 show the $L(h,k)$ labeling of cycles.

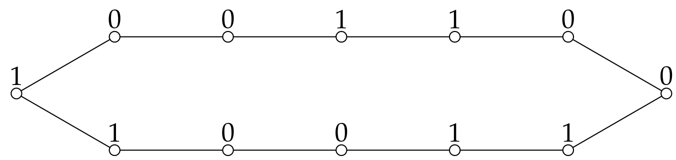

Figure 11: $L(0,1)$ labeling of C_{12}

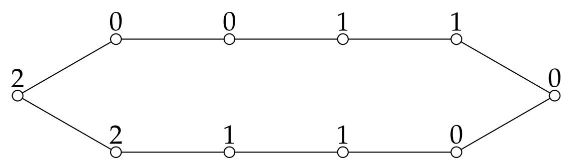

Figure 12: $L(0,1)$ labeling of C_{10}

Theorem 2.5. [7] *For any h and k we have .*

$$\lambda_{h,k}(C_n) = \begin{cases} 2h & \begin{array}{l} \textit{if } n = \textit{odd} \text{ , } n \geq 3 \textit{ and } h \leq 2k \textit{ or} \\ \textit{if } n \equiv 0 \pmod{3} \textit{ and } h \geq 2k \end{array} \\ h{+}2k & \begin{array}{l} \textit{if } n \equiv 0 \pmod{4} \textit{ and } h \geq 2k \textit{ or} \\ \textit{if } n \not\equiv 0 \pmod{3}, n \neq 5 \textit{ and } h \leq 2k \end{array} \\ 2h, & \textit{if } n \equiv 2 \pmod{4} \textit{ and } h \leq 3k \\ h+3k, & \textit{if } n \equiv 2 \pmod{4} \textit{ and } h \geq 3k \\ 2h, & \textit{if } n \geq 5 \textit{ and } h \leq 2k \\ 4k, & \textit{if } n = 5 \end{cases}$$

Theorem 2.6. [8, 9] *Consider the graph $G \equiv K_n \square K_m$. When $2 \leq m < n$, we have the following result.*

$$\lambda_{h,k}(G) = \begin{cases} (m-1)h + (n-1)k, & \textit{for } \frac{h}{k} > n \\ (mn-1)k, & \textit{for } \frac{h}{k} \geq n \\ (n-1)h + (2n-1)k, & \textit{for } \frac{h}{k} > (n-1) \\ (n^2-1)h, & \textit{for } \frac{h}{k} < (n-1) \end{cases}$$

In 2003, Erwin et al. gave the $L(h,k)$ labeling for the Cartesian product of complete graphs [6].

Theorem 2.7. [6] *Consider the graph $G = \prod_{i=1}^{n} K_{t_i}$ which is the Cartesian product of n complete graphs .*

When $n > 3$ and all the t_i's are relatively prime when $2 \le t_1 < t_2 < t_3 < \ldots < t_n$ we have,

$$\lambda_{h,k}(G) = \begin{cases} (t_n t_{n-1} - 1)k, & \text{for } \frac{h}{k} \le t_{n-1} \\ (t_n - 1)h + (t_{n-1} - 1)k, & \text{for } \frac{h}{k} > t_{n-1} \end{cases}$$

In 2009, Huang et al [5], gave a potential labeling for the graph $G = \prod_{i=1}^{n} K_{n_i}$, where there is no restriction on n_i which provides an upper bound, where the notation $\prod_{i=1}^{n} G_{n_i}$ represents Cartesian product of the graphs.

Theorem 2.8. [2] *Bertossi et al. proved that $L(0,1)$ labeling for binary trees requires only 3 colors.*

Theorem 2.9. [2] $\lambda_{1,1}(B) = \Delta$, *where B is a binary tree*

A *Tree* is an undirected graph in which any two vertices are connected by exactly one path.

Theorem 2.10. [12] $(\Delta + 1) \le \lambda_{2,1}(T) \le (\Delta + 2)$, *where T is a tree.*

Later on, Chang and Kuo gave a polynomial time algorithm based on dynamic programming to find the span .

3. $L(h,k)$ LABELING OF $K_n \square P_m$

We start by proving the following lemma.

Lemma 3.1. *Let H be an induced subgraph of G. Then $\lambda_{(h,k)}(H) \leq \lambda_{(h,k)}(G)$.*

Proof. Let f be an optimal $L(h,k)$ labeling of G. Consider two vertices u and v such that $u, v \in V(H)$. If $d_H(u,v) = 1$ then $d_G(u,v) = 1$. Therefore we have $|f(u) - f(v)| \geq h$. If $d_H(u,v) = 2$, then $d_G(u,v) = 2$ (since H is an induced subgraph of G). Therefore $|f(u) - f(v)| \geq k$. Therefore f restricted to H is an $L(h,k)$ labeling for H. Hence we have $\lambda_{h,k}(H) \leq \lambda_{h,k}(G)$. □

Remark 1. The proof does not work if H is not an induced subgraph of G. For example consider the graphs K_4 and its subgraph C_4.

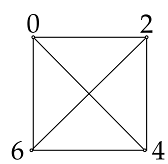

Figure 13: $L(2,6)$ labeling of K_4

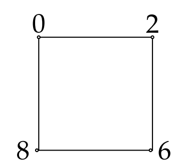

Figure 14: $L(2,6)$ labeling of C_4

From Figures 13 and 14, we have $\lambda_{2,6}(K_4) < \lambda_{2,6}(C_4)$.

Remark 2. *If $h \geq k$, and if H is a subgraph of G, then $\lambda_{h,k}(H) \leq \lambda_{h,k}(G)$.*

3.1 $h \geq 2k$

Theorem 3.1. *$\lambda_{h,k}(K_n \square P_m) = (n-1)h + k$, when $h \geq 2k$.*

Proof. Consider the graph H which is $K_n \square P_2$. Let f be an optimal $L(h,k)$ labeling of H with the highest label less than or equal to $(n-1)h + k - 1$. Without loss of generality let the highest label be $f(v_{x,1})$. Any vertex $v_{j,2}$ in the second layer will be either at a distance of one or two from $v_{x,1}$. Consider the vertex $v_{x,2}$ in the second layer which is at a distance of one from $v_{x,1}$, therefore:

$$\begin{aligned} f(v_{x,2}) &\leq (n-1)h + k - 1 - h \\ &= (n-2)h + k - 1 \\ &< (n-1)h \end{aligned}$$

Consider any vertex $v_{j,2}$ in the second layer which is at a distance of two from $v_{x,1}$, then

$$\begin{aligned} f(v_{j,2}) &\leq (n-1)h+k-1-k \\ &= (n-1)h-1 \\ &< (n-1)h \end{aligned}$$

which is a contradiction to Observation 1.

Hence we have $\lambda(H) \geq (n-1)h+k$. Since the graph H is an induced subgraph of the graph $K_n \Box P_m$, we have, $\lambda(K_n \Box P_m) \geq (n-1)h+k$. Also we have a labeling which uses $(n-1)h+k$ labels. The labeling is as follows.

In the j^{th} layer, when j is odd, any vertex $v_{i,j}$ will have label $(i-1-\frac{j-1}{2})h \pmod{nh}$ for $0 \leq i \leq n-1$.

In the j^{th} layer, when j is even, any vertex $v_{i,j}$ will have label $(i+1-\frac{j}{2})h+k \pmod{nh}$ for $0 \leq i \leq n-1$ respectively.

Now we give the proof that this an $L(h,k)$ labeling

Case 1. In the j^{th} layer, when j is odd, we have $f(v_{i,j}) = (i-1-\frac{j-1}{2})h \pmod{nh}$.

If we take two vertices in the same layer then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j})| &= |(i_1-1-\tfrac{j-1}{2})h - (i_2-1-\tfrac{j-1}{2})h \pmod{nh}| \\ &= |(i_1-i_2)h| \\ &\geq h \end{aligned}$$

If we take a vertex from $(j+1)^{st}$ layer then $|f(v_{i_1,j}) - f(v_{i_2,j+1})| = |(i_1-1-\frac{j-1}{2})h - (i_2+1-\frac{j+1}{2})h - k \pmod{nh}|$. If $i_1 = i_2$, then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |-h+k \pmod{nh}| \\ &> h \end{aligned}$$

If $i_1 \neq i_2$ then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |(i_1-i_2-1)h-k \pmod{nh}| \\ &\geq |k \pmod{nh}| \end{aligned}$$

If we take a vertex from $(j+2)^{nd}$ layer only $v_{i,j+2}$ will be at a distance of two. So

$$\begin{aligned} |f(v_{i,j}) - f(v_{i,j+2})| &= |(i_1-1-\tfrac{j-1}{2})h - (i_1-1-\tfrac{j+1}{2})h \pmod{nh}| \\ &= |h \pmod{nh}| \\ &> k. \end{aligned}$$

Case 2. *In the j^{th} layer, when j is even, we have $f(v_{i,j}) = (i+1-\frac{j}{2})h + k \pmod{nh}$.*
If we take two vertices in the same layer then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j})| &= |(i_1+1-\tfrac{j}{2})h + k - (i_2+1-\tfrac{j}{2})h - k \pmod{nh}| \\ &= |(i_1-i_2)h| \\ &\geq h \end{aligned}$$

If we take a vertex from $(j+1)^{th}$ layer then $|f(v_{i_1,j}) - f(v_{i_2,j+1})| = |(i_1+1-\frac{j}{2})h + k - (i_2-1-\frac{j}{2})h \pmod{nh}|$. If $i_1 = i_2$, then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |2h + k \pmod{nh}| \\ &> h \end{aligned}$$

If $i_1 \neq i_2$ then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |(i_1-i_2+2)h + k| \pmod{nh} \\ &\geq k (\text{ since } k < h) \end{aligned}$$

If we take a vertex from $(j+2)^{nd}$ layer only $v_{i,j+2}$ will be at a distance of two. So

$$\begin{aligned} |f(v_{i,j}) - f(v_{i,j+2})| &= |(i+1-\tfrac{j}{2})h + k - (i+1-\tfrac{j+2}{2})h - k \pmod{nh}| \\ &= |-h \pmod{nh}| \\ &= (n-1)h \\ &> k. \end{aligned}$$

□

We show an $L(5,2)$ labeling for $K_6 \square P_4$ in Figure 15.

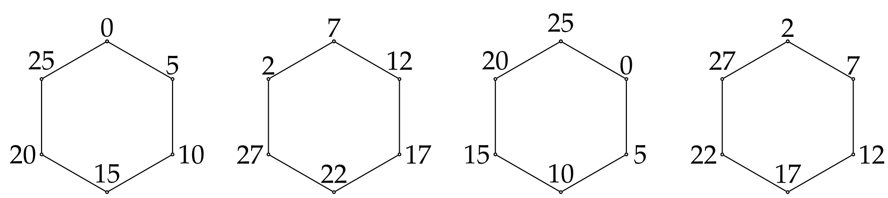

Figure 15: $L(5,2)$ labeling of $K_6 \square P_4$

Remark 3. The labeling does not work when $h < 2k$. For example if $h = 5$ and $k = 3$ when we label the graph $K_6 \square P_2$ using the above mentioned labeling as shown in Figure 16, we have a conflict since the vertex which has label 8 is at a distance of two from the vertex which has label 10 and $k = 3$. Hence the distance two criteria is not satisfied.

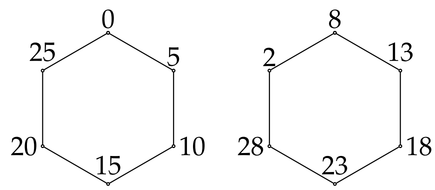

Figure 16: Labeling in Theorem 3.1 fails for $L(5,3)$

3.2 $2k > h \geq k$

Theorem 3.2. *If* $2k \geq h \geq k$, *then* $\lambda(K_n \square P_m) = (2n-1)k$.

Proof. Consider the graph H which is $K_n \square P_2$. Let f be an optimal $L(h,k)$ labeling of H. In the graph H each vertex is either at a distance of one or two from every other vertex. Since $h > k$, we have the highest label required is at least $(2n-1)k$. Therefore $\lambda(H) \geq (2n-1)k$.
Since H is an induced subgraph of $K_n \square P_m$ we have $\lambda(K_n \square P_m) \geq (2n-1)k$.
Also we have an $L(h,k)$ labeling of the graph H with $(2n-1)k$ as the highest label. We use the labeling which was explained in the proof of Theorem 3.1 with $h = 2k$. $\square$

We show $L(6,4)$ labeling of $K_6 \square P_4$ in Figure 17.

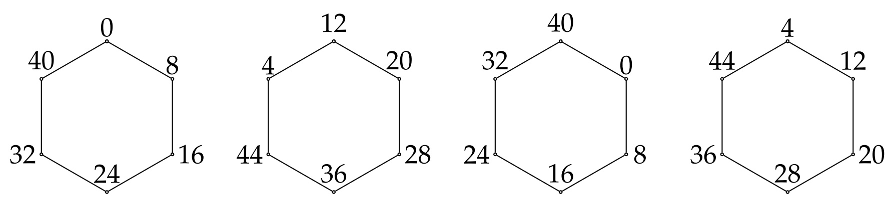

Figure 17: $L(6,4)$ labeling for $K_6 \square P_4$

Remark 4. Using a brute force program we checked that although the labeling works when $h < k$ it is no longer an optimal $L(h,k)$ labeling. Consider two $L(4,8)$ labelings of the graph $K_6 \square P_4$ shown in Figures 18 and 19.

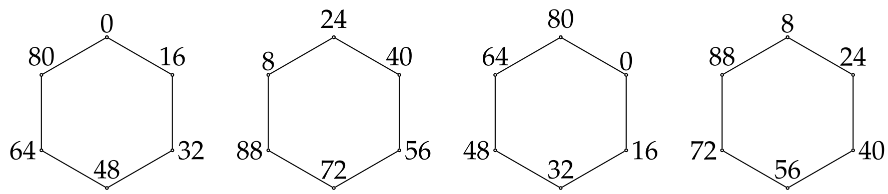

Figure 18: $L(4,8)$ labeling for $K_6 \square P_4$ using the above mentioned labeling

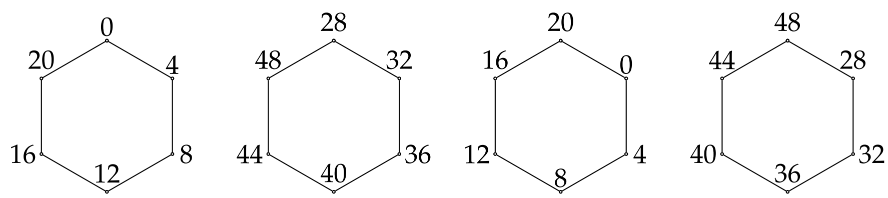

Figure 19: $L(4,8)$ labeling for $K_6 \square P_4$ found using the brute force program

From Figures 18 and 19 we can say that labeling fails when $h < k$

3.3 $h < k \leq \lfloor \frac{n}{2} \rfloor h$

Theorem 3.3. *If $h < k \leq \frac{nh}{2}$ and $m, n \geq 3$, then $\lambda(K_n \square P_m) = 2(n-1)h + k$.*

Proof. Consider the graph $H = K_n \square P_3$. Let f be an optimal $L(h,k)$ labeling of H. Assume that the highest label is less than or equal to $2(n-1)h + k - 1$. Now consider the first two layers. There are $2n$ vertices. Let us arrange the vertices in the increasing order of their labeling. That is $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. The vertex v_i can either belong to the first layer or the second. Let x_i be the difference between $f(v_i)$ and $f(v_{i+1})$. Since each vertex is either at a distance of one or two from any other vertex we have $x_i \geq h$.

<u>Claim:</u> $x_1 + x_2 + x_3 + \ldots + x_{2n-1} \leq 2(n-1)h + k - 1$.

<u>Proof of the claim:</u> We know that $x_1 = f(v_2) - f(v_1)$, $x_2 = f(v_3) - f(v_2)$. Similarly, $x_l = f(v_{l+1}) - f(v_l)$, for any l such that $1 \leq l \leq 2n-1$ So when we add all x_i's we have their sum equal to $f(v_{2n}) - f(v_1)$. We know that $f(v_1) \geq 0$ and $f(v_{2n}) \leq 2(n-1)h + k - 1$. Hence we have $x_1 + x_2 + x_3 + \ldots + x_{2n-1} \leq 2(n-1)h + k - 1$. ∎

<u>Claim:</u> $x_i \leq k - 1$, for $1 \leq i \leq 2n-1$

<u>Proof of the claim:</u> Assume for some j, we have $x_j \geq k$. We still have $(2n-2)$ other x_i's and each $x_i \geq h$. Their sum $x_1 + x_2 + x_3 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_{2n-1} \geq (2n-2)h$, adding x_j to the sum

we have

$$\begin{aligned}x_1+x_2+x_3+\ldots+x_{j-1}+x_j+x_{j+1}+\ldots+x_{2n-1} &\geq (2n-2)h+k\\ &= 2(n-1)h+k\\ &> 2(n-1)h+k-1\end{aligned}$$

which is a contradiction. Therefore $h \leq x_i \leq k-1$. ■

Now consider the first and second layers, there has to be at least one transition between the layers. Let us assume that there is a transition at vertex v_i. That is the vertex v_i will belong to one layer and v_{i+1} will belong to the layer adjacent to it. There are two cases that we need to consider. We first prove a claim

Claim: If $h < k \leq \frac{nh}{2}$ and $n > 3$, then we have $\lambda(K_n \square P_2) \geq (2n-3)h+k$

Proof of the claim: If we have a combined 2-transition between the layers then we come across set of vertices v_l, v_{l+1}, v_{l+2} such that v_l and v_{l+2} belong to one layer and v'_{l+1} belongs to an other layer. Without loss of generality assume that the vertices v_l and v_{l+2} are in second layer and v_{l+1} is in first layer. Now the vertex v_{l+1} cannot be adjacent to both v_l and v_{l+2}, hence either $d(v_l, v_{l+1}) = 2$ or $d(v_{l+2}, v_{l+1}) = 2$. Therefore either $x_l \geq k$ or $x_{l+1} \geq k$, which is a contradiction to the above mentioned claim. So only isolated transitions exist. Consider an isolated transition at some vertex v_j. Assume that the vertex v_j is in second layer and v_{j+1} is in first layer. We know that either v_{j-1} or v_{j+2} exist. So either $x_{j-1}+x_j \geq k$ (since $d(v_{j-1}, v_{j+1}) = 2$) or $x_j + x_{j+1} \geq k$ (since $d(v_j, v_{j+2}) = 2$). Also we know that $x_i \geq h$ for any i, therefore $x_1+x_2+x_3+\ldots+x_j+x_{j+1}+x_{j+2}+\ldots+x_{2n-2}+x_{2n-1} \geq (2n-3)h+k$ ■

Case 1. Let us assume that in the second layer we come across a vertex v_{i+1} such that $f(v_{i+1}) = y$ and in the first layer we have the vertex v_i such that $f(v_i) = y - x_i$.

Since $x_i \leq k-1$, the distance between both the vertices should be one. That is $d(v_i, v_{i+1}) = 1$. The vertices are shown in Figure 20.

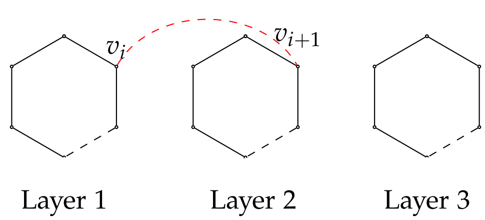

Figure 20: $K_n \square P_3$

Now consider the second and third layers. We again have $2n$ vertices with the same restrictions. When we arrange the vertices in the increasing order of their labels and take the differences we have an another set of $x'_1, x'_2, x'_3 \ldots x'_{2n-1}$. The vertex v_{i+1} is in the second layer. So now consider the vertex which comes before the vertex v_{i+1} after arranging the vertices in the increasing order. Let that vertex be v'_i. Therefore we have $f(v'_i) = y - x'_l$.
If the vertex v'_i is in the second layer, then we have $d(v_i, v'_i) = 2$ as shown in Figure 21.

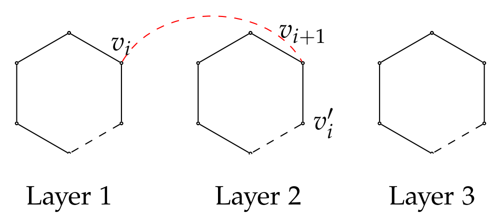

Figure 21: $K_n \square P_3$

If we calculate the difference between their labels we have $|f(v_i) - f(v'_i)| = |x_i - x'_l| < (k-1) < k$. If the vertex v'_i is in the third layer and if $d(v_{i+1}, v'_i) = 2$ as shown in Figure 22,then we have $|f(v_{i+1}) - f(v'_i)| = |x'_i| \leq (k-1) < k$, a contradiction.

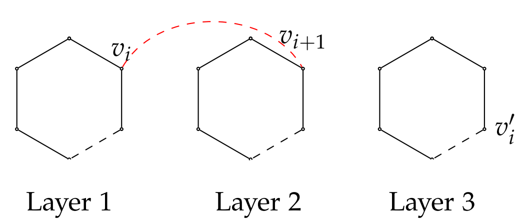

Figure 22: $K_n \square P_3$

If $d(v_{i+1}, v'_i) = 1$ as shown in Figure 23, then We have $d(v_i, v'_i) = 2$. Therefore $|f(v_i) - f(v'_i)| \geq k$, but if we calculate the difference between their labels we have $|f(v_i) - f(v'_i)| = |x_i - x'_i| < (k-1) < k$.

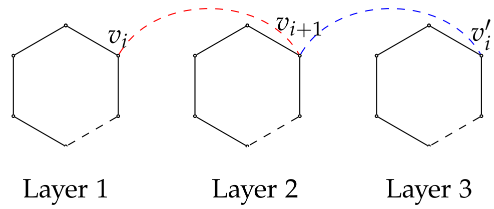

Figure 23: $K_n \square P_3$

Now if v_{i+1} is the vertex with lowest label in second and third layers, since $f(v_i) < f(v_{i+1})$, we have $f(v_{i+1}) \geq h$. By the above claim we can say that the difference between highest and lowest labels in the second and third layers should be greater than or equal to $(2n-3)h+k$. Now if v'_{2n} is the vertex with highest label in second and third layers then $f(v'_{2n}) \geq (2n-3)h+k+f(v_{i+1})$. Therefore $f(v'_{2n}) \geq (2n-2)h+k$.

Case 2. Now we assume that we come across a vertex v_i in the second layer and the vertex v_{i+1} is in the first layer with $f(v_i) = y$ and $f(v_{i+1}) = y + x_i$.
Since $x_i \leq k-1$, we have $d(v_i, v_{i+1}) = 1$ as shown in Figure 24.

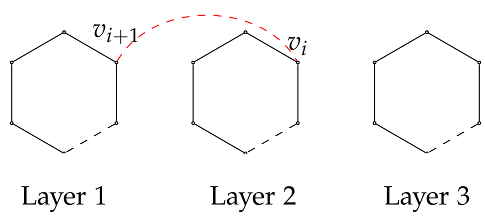

Figure 24: $K_n \square P_3$

Now consider the second and third layers. We again have $2n$ vertices with same restrictions. When we arrange the vertices in increasing order of their labels and take their differences we have $x'_1, x'_2, x'_3, \ldots, x'_{2n-1}$. Consider the vertex v'_{i+1} which comes immediately after the vertex v_i when arranged in the increasing order. The vertex will have the label $y + x'_i$.
If the vertex v'_{i+1} is in the second layer as shown in Figure 25, then $d(v_{i+1}, v'_{i+1}) = 2$, but if we calculate the difference between the labels we have $|f(v_{i+1}) - f(v'_{i+1})| = |x_i - x'_l| < (k-1) < k$.

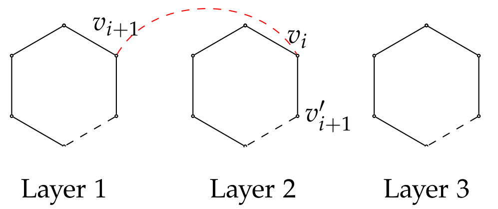

Figure 25: $K_n \square P_3$

If the vertex v'_{i+1} is in the third layer and if $d(v_i, v'_{i+1}) = 2$ as shown in Figure 26, then we have $|f(v'_{i+1}) - f(v_i)| = |x'_i| \leq (k-1) < k$.

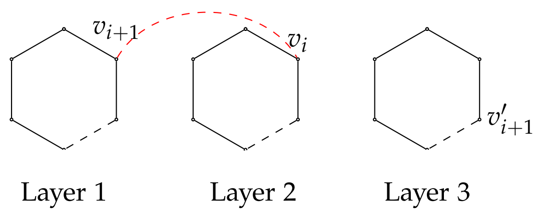

Figure 26: $K_n \square P_3$

If $d(v'_{i+1}, v_i) = 1$, as shown in Figure 27, then we have $d(v_{i+1}, v'_{i+1}) = 2$, which again gives us $|f(v_{i+1}) - f(v'_i)| = |x_i - x'_i| \leq (k-1) < k$.

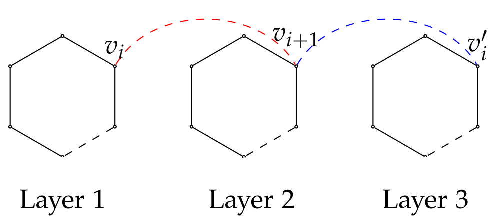

Figure 27: $K_n \square P_3$

If v_i is vertex with highest label in second and third layers then v'_{i+1} does not exist. Since v_i is the vertex with highest label we have $f(v_i) \geq (2n-3)h + k$ from Claim 3. So $f(v_{i+1}) \geq$

$(2n-3)h+k+h$

$$\begin{aligned} f(v_{i+1}) &\geq (2n-3)h+k+h \\ &= (2n-2)h+k \\ &= 2(n-1)h+k \end{aligned}$$

Hence f cannot be an $L(h,k)$ labeling of the graph.

Therefore, $\lambda(K_n \Box P_3) \geq 2(n-1)h+k$. Since H is an induced subgraph of $K_n \Box P_m$ for $m \geq 3$ we have $\lambda(K_n \Box P_m) \geq 2(n-1)h+k$. We now describe the labeling with highest label as $2(n-1)h+k$.

The labeling.

For the j^{th} layer:

when $j \equiv 1 \pmod 4$, for any vertex $v_{i,j}$, we have $f(v_{i,j}) \equiv (i-1)h \pmod{nh}$, for $1 \leq i \leq n$.

when $j \equiv 2 \pmod 4$, for any vertex $v_{i,j}$, we have $f(v_{i,j}) \equiv (i-1)h \pmod{nh} + (n-1)h+k$, for $1 \leq i \leq n$.

when $j \equiv 3 \pmod 4$, for any vertex $v_{i,j}$, we have $f(v_{i,j}) \equiv (\lfloor \frac{n}{2} \rfloor + i - 1)h \pmod{nh}$, for $1 \leq i \leq n$.

when $j \equiv 0 \pmod 4$, for any vertex $v_{i,j}$, we have $f(v_{i,j}) \equiv (\lfloor \frac{n}{2} \rfloor + i - 1)h \pmod{nh} + (n-1)h+k$, for $1 \leq i \leq n$.

Now we give the proof of labeling.

Case 1. In j^{th} layer when $j \equiv 1 \pmod 4$ we have $f(v_{i,j}) \equiv (i-1)h \pmod{nh}$.

If we take two vertices in the same layer then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j})| &= |(i_1-1)h - (i_2-1)h \pmod{nh}| \\ &= |(i_1-i_2)h| \\ &\geq h \end{aligned}$$

If we take a vertex from $(j+1)^{th}$ layer then $|f(v_{i_1,j}) - f(v_{i_2,j+1})| = |(i_1-1)h - (i_2-1)h \pmod{nh} - (n-1)h+k|$. If $i_1 = i_2$, then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |0 \pmod{nh} + (n-1)h+k| \\ &> h \end{aligned}$$

If $i_1 \neq i_2$ then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |(i_1-i_2)h \pmod{nh} - ((n-1)h+k)| \\ &\geq k(\text{since } (i_1-i_2)h \pmod{nh} \leq ((n-1)h) \end{aligned}$$

If we take a vertex from $(j+2)^{th}$ layer only $v_{i,j+2}$ will be at a distance of two. So

$$\begin{aligned} |f(v_{i,j}) - f(v_{i,j+2})| &= |(i-1)h - (\lfloor \tfrac{n}{2} \rfloor + i - 1))h \pmod{nh}| \\ &= |-\lfloor \tfrac{n}{2} \rfloor h \pmod{nh}| \\ &= \lfloor \tfrac{n}{2} \rfloor h \\ &\geq k \text{ (since } k \leq \lfloor \tfrac{n}{2} \rfloor h). \end{aligned}$$

Case 2. In j^{th} layer when $j \equiv 2 \pmod 4$ we have $f(v_{i,j}) \equiv (i-1)h \pmod{nh} + (n-1)h + k$.

If we take two vertices in the same layer then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j})| &= |(i_1-1)h - (i_2-1)h \pmod{nh} + (n-1)h + k - ((n-1)h+k)| \\ &= |(i_1 - i_2)h| \\ &\geq h \end{aligned}$$

If we take a vertex from $(j+1)^{th}$ layer then $| f(v_{i_1,j}) - f(v_{i_2,j+1}) = (i_1-1)h - (\lfloor \frac{n}{2} \rfloor + i_2 - 1)h \pmod{nh} + (n-1)h + k|$. If $i_1 = i_2$, then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |-\lfloor \tfrac{n}{2} \rfloor h \pmod{nh} + (n-1)h + k| \\ &= \lfloor \tfrac{n}{2} \rfloor h + (n-1)h + k \\ &> h. \end{aligned}$$

If $i_1 \neq i_2$ then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= (-\lfloor \tfrac{n}{2} \rfloor + i_1 - i_2)h \pmod{nh} + ((n-1)h + k)| \\ &\geq k \text{ (since } (i_1 - i_2)h \pmod{nh} - \lfloor \tfrac{n}{2} \rfloor \leq ((n-1)h) \end{aligned}$$

If we take a vertex from $(j+2)^{th}$ layer only $v_{i,j+2}$ will be at a distance of two. So

$$\begin{aligned} |f(v_{i,j}) - f(v_{i,j+2})| &= |(i - 1 - \lfloor \tfrac{n}{2} \rfloor - i + 1))h \pmod{nh} + (n-1)h + k| \\ &= |-\lfloor \tfrac{n}{2} \rfloor h \pmod{nh} + (n-1)h + k| \\ &= \lfloor \tfrac{n}{2} \rfloor h + (n-1)h + k \\ &\geq k. \end{aligned}$$

Case 3. In j^{th} layer when $j \equiv 3 \pmod 4$ we have $f(v_{i,j}) \equiv (\lfloor \frac{n}{2} \rfloor + i - 1)h \pmod{nh}$.

If we take two vertices in the same layer then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j})| &= |(\lfloor \tfrac{n}{2} \rfloor + i_1 - 1)h - (\lfloor \tfrac{n}{2} \rfloor + i_2 - 1)h \pmod{nh}| \\ &= |(i_1 - i_2)h| \\ &\geq h \end{aligned}$$

If we take a vertex from $(j+1)^{th}$ layer then $|f(v_{i_1,j}) - f(v_{i_2,j+1})| = |(\lfloor \frac{n}{2} \rfloor + i_1 - 1)h - (\lfloor \frac{n}{2} \rfloor + i_2 - 1)h \pmod{nh} - (n-1)h + k |$. If $i_1 = i_2$, then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |0 \pmod{nh} - (n-1)h + k| \\ &> h \end{aligned}$$

If $i_1 \neq i_2$ then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |(i_1 - i_2)h \pmod{nh} - ((n-1)h + k)| \\ &\geq k \end{aligned}$$

If we take a vertex from $(j+2)^{th}$ layer only $v_{i,j+2}$ will be at a distance of two. So

$$\begin{aligned} |f(v_{i,j}) - f(v_{i,j+2})| &= |(\lfloor \tfrac{n}{2} \rfloor + i - 1 - i + 1)h \pmod{nh}| \\ &= |\lfloor \tfrac{n}{2} \rfloor h \pmod{nh}| \\ &= |\lfloor \tfrac{n}{2} \rfloor h| \\ &\geq k \text{ (since } k \leq \lfloor \tfrac{n}{2} \rfloor h). \end{aligned}$$

Case 4. In j^{th} layer when $j \equiv 0 \pmod 4$ we have $f(v_{i,j}) \equiv (\lfloor \frac{n}{2} \rfloor + i - 1)h \pmod{nh}$+(n-1)h+k.
If we take two vertices in the same layer then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j})| &= |(\lfloor \tfrac{n}{2} \rfloor + i_1 - 1)h - (\lfloor \tfrac{n}{2} \rfloor + i_2 - 1)h \pmod{nh} + (n-1)h + k - ((n-1)h + k)| \\ &= |(i_1 - i_2)h| \\ &\geq h \end{aligned}$$

If we take a vertex from $(j+1)^{th}$ layer then $|f(v_{i_1,j}) - f(v_{i_2,j+1})| = |(\lfloor \frac{n}{2} \rfloor + i_1 - 1)h - (\lfloor \frac{n}{2} \rfloor + (i_2 - 1)h \pmod{nh} + (n-1)h + k|$. If $i_1 = i_2$, then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |(\lfloor \tfrac{n}{2} \rfloor + i - 1 - i + 1) \pmod{nh} + (n-1)h + k| \\ &> h \end{aligned}$$

If $i_1 \neq i_2$ then

$$\begin{aligned} |f(v_{i_1,j}) - f(v_{i_2,j+1})| &= |(i_1 - i_2)h \pmod{nh} + ((n-1)h + k)| \\ &\geq k \end{aligned}$$

If we take a vertex from $(j+2)^{th}$ layer only $v_{i,j+2}$ will be at a distance of two. So

$$\begin{aligned} |f(v_{i,j}) - f(v_{i,j+2})| &= |(\lfloor \tfrac{n}{2} \rfloor + i - 1 - i + 1)h \pmod{nh}| \\ &= |\lfloor \tfrac{n}{2} \rfloor h \pmod{nh}| \\ &= \lfloor \tfrac{n}{2} \rfloor h \\ &\geq k \text{ (since } k \leq \lfloor \tfrac{n}{2} \rfloor h). \end{aligned}$$

$\square$

In Figure 28 we show $L(5,9)$ labeling of $K_6 \square P_4$.

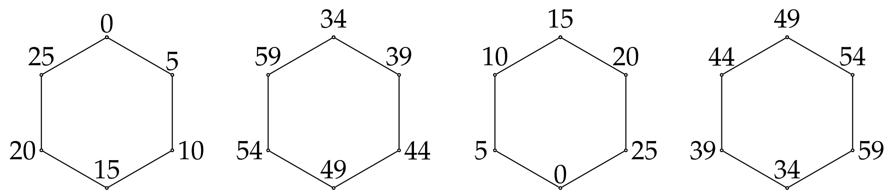

Figure 28: $L(5,9)$ labeling for $K_6 \square P_4$

Remark 5. The labeling does not work when $k > \lfloor \frac{n}{2} \rfloor h$. Using a brute force algorithm we checked that to get a general expression for the optimal $L(h,k)$ labeling, we need to analyze five layers of the complete graph.

3.4 $\lfloor \frac{n}{2} \rfloor h < k \leq (n-1)h$

Lemma 3.4. *Let f be an optimal $L(h,k)$ labeling of the graph $K_n \square P_2$ with $\frac{nh}{2} < k \leq (n-1)h$, where n is an even number greater than 4 and the vertex with zero label is in the first layer. If the highest label in the first layer is less than $(\frac{n}{2}-1)h+k$ then every vertex in the second layer must have label greater than or equal to $(n-2)h+k$.*

Proof. Consider the graph $K_n \square P_2$ and arrange the vertices of the first layer in the increasing order of their labeling. That is, $f(v_{1,1}) < f(v_{2,1}) < f(v_{3,1}) < \ldots < f(v_{i,1}) < f(v_{i+1,1}) < \ldots < f(v_{n,1})$. Now let us suppose we can use some value between $f(v_{i,1})$ and $f(v_{i+1,1})$ to label some vertex $v_{j,2}$ in the second layer, so we have $f(v_{i,1}) < f(v_{j,2}) < f(v_{i+1,1})$, where $v_{i,1}$ is the i^{th} highest labeled vertex in the first layer. Now if $v_{j,2}$ is adjacent to $v_{i,1}$ then $f(v_{j,2}) \geq f(v_{i,1}) + h$, and $v_{j,2}$ is at a distance of two from $v_{i+1,1}$. So

$$\begin{aligned} f(v_{i+1,1}) &\geq f(v_{j,2}) + k \\ &\geq f(v_{i,1}) + h + k \end{aligned}$$

Similarly if $v_{j,2}$ is adjacent to $v_{i+1,1}$ then $f(v_{j,2}) \leq f(v_{i+1,1}) - h$ and the vertex $v_{j,2}$ will be at a distance of two from $v_{i,1}$, therefore

$$\begin{aligned} f(v_{i,1}) &\leq f(v_{j,2}) - k \\ &\leq f(v_{i+1,1}) - h - k. \end{aligned}$$

If $v_{j,2}$ is not adjacent to either $v_{i,1}$ or $v_{i+1,1}$ then $f(v_{j,2}) \geq f(v_{i,1}) + k$ and $f(v_{i+1,1}) \geq f(v_{j,2}) + k$. Thus $f(v_{i+1,1}) \geq f(v_{i,1}) + k + k$.
Therefore in any case we have $f(v_{i+1,1}) - f(v_{i,1}) \geq h + k$. Now in the first layer we have $f(v_{i,1}) \geq$

$(i-1)h$. Therefore $f(v_{i+1,1}) \geq (i-1)h + h + k$.

We still have $n-(i+1)$ vertices in the first layer with labels greater than $f(v_{i+1,1})$. Therefore $f(v_{n,1}) \geq ih + k + (n-i-1-1)h = (n-2)h + k > (\frac{n}{2}-1)h + k$ for $n > 2$. But the highest label in the first layer is less than $(\frac{n}{2}-1)h + k$. Therefore we cannot use any value between 0 and $f(v_{n,1})$ for any vertex in the second layer.

Now, we know that $f(v_{n,1} \geq (n-1)h$. Let the lowest labeled vertex in the second layer be $v_{1,2}$. If $v_{1,2}$ is at a distance of two from $v_{n,1}$ then $f(v_{1,2}) \geq (n-1)h + k$. If $v_{1,2}$ is adjacent to $v_{n,1}$, then it will be at a distance of two from the $v_{n-1,1}$. Therefore

$$\begin{aligned} f(v_{1,2}) &\geq f(v_{n-1,1}) + k \\ &\geq (n-2)h + k. \end{aligned}$$

□

Lemma 3.5. *Let f be an optimal $L(h,k)$ labeling of $K_n \square P_3$, where $\frac{nh}{2} \leq k \leq (n-1)h$ and n is an even number greater than 4. If the highest label used in the first layer is less than $(\frac{n}{2}-1)h + k$ then we have highest label in third layer greater than or equal to $(\frac{n}{2})h + k$ and second highest label greater than or equal to $(\frac{n}{2}-1)h + k$.*

Proof. Let us arrange the vertices of the first layer in increasing order of their labeling, so $f(v_{1,1}) < f(v_{2,1}) < f(v_{3,1}) < \ldots < f(v_{n-1,1}) < f(v_{n,1})$. Consider the vertex $v_{\frac{n}{2}+1,1}$, we know that $f(v_{\frac{n}{2}+1,1}) \geq (\frac{n}{2})h$. If $f(v_{\frac{n}{2},1}) \geq k$, then we will have $f(v_{n,1}) \geq k + (\frac{n}{2}-1)h$, which is a contradiction. Therefore $f(v_{\frac{n}{2}+1,1}) < k$. In the third layer let the vertex which is at a distance of two from $v_{\frac{n}{2}+1,1}$ be $v_{\frac{n}{2}+1,3}$. We know that $|f(v_{\frac{n}{2}+1,3}) - f(v_{\frac{n}{2}+1,1})| \geq k$. Since $f(v_{\frac{n}{2}+1,1}) < k$, we have $f(v_{\frac{n}{2}+1,3}) \geq f(v_{\frac{n}{2}+1,1}) + k$. Therefore $f(v_{\frac{n}{2}+1,3}) \geq (\frac{n}{2})h + k$.

Similarly $f(v_{\frac{n}{2},1}) \geq (\frac{n}{2}-1)h$, so $f(v_{\frac{n}{2},3}) \geq f(v_{\frac{n}{2},1}) + k \geq (\frac{n}{2}-1)h + k$. □

Lemma 3.6. *Let f be an $L(h,k)$ labeling of $K_n \square P_3$, where $\frac{nh}{2} \leq k \leq (n-1)h$, n is an even number greater than 4 and there is only one transition between any two adjacent layers. If the lowest label in the first layer is less than the lowest label in the second layer which is less than the lowest label in the third layer, then we have highest label under f is greater than or equal to $(n-2)h + 3k$.*

Proof. Consider the three layers as shown in Figure 29.

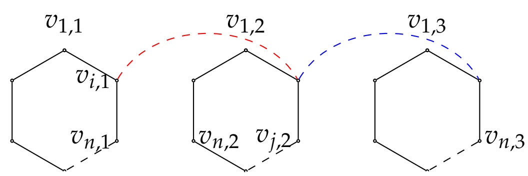

Figure 29: $K_n \square P_3$

Let $v_{1,1}$, $v_{1,2}$, $v_{1,3}$ be the vertices with the lowest label in the first layer, second layer and the third layer respectively. By assumption we know that $f(v_{1,1}) < f(v_{1,2})$. Now suppose there are vertices $v_{i,1}$ and $v_{j,2}$ such that $f(v_{i,1}) > f(v_{j,2})$. Then will have at least two transitions between layers one and two, since we have $f(v_{1,1}) < f(v_{2,1}) < f(v_{j,2}) < f(v_{i,1})$ violating our assumption. Therefore no vertex in the first layer has label greater than $f(v_{1,2})$. Similarly no vertex in the second layer has a label greater than $v_{1,3}$.

Let $v_{n,1}$ be the vertex with highest label in the first layer. Then we have $f(v_{n,1}) \geq (n-1)h$. There are at least $n-1$ vertices in the second layer that are at a distance of two from $v_{n,1}$. Therefore if $v_{y,2}$ is the vertex with the highest label in the second layer, we have $f(v_{y,2}) \geq (n-1)h + k + (n-2)h$. There are $n-1$ vertices in the third layer that are at a distance of two from $v_{y,2}$. Therefore if $v_{n,3}$ is the vertex with the highest label in the third layer, we have $f(v_{n,3}) \geq (n-1)h + k + (n-2)h + k + (n-2)h \geq 3k + (n-2)h$. □

Lemma 3.7. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$, where $\frac{nh}{2} < k \leq (n-1)h$, where n is an even number greater than 4 and there are two transition between the layers. Assume that the vertex with the smallest label is in the second layer and let that vertex be $v_{1,2}$. Let $v_{1,1}$ be the vertex with the smallest label in the first layer. Let $v_{n,1}$ be the vertex with highest label in the first layer. Suppose there are x vertices in the second layer that are at a distance of two from $v_{1,1}$ and that have label lower than $f(v_{1,1})$ and there are y vertices in the second layer that are at a distance of two from $v_{n,1}$ and that have label higher than $f(v_{n,1})$. Then we have $x + y \geq n - 2$.*

Proof. Since we have only two transitions between the layers any vertex in the second layer will have label either less than $f(v_{1,1})$ or greater than $f(v_{n,1})$. Let $v_{1,2}$ be the vertex in the second layer adjacent to $v_{1,1}$ and let $v_{n,2}$ be the vertex in the second layer adjacent to $v_{n,1}$. If $f(v_{1,2}) < f(v_{1,1})$ and $f(v_{n,2}) > f(v_{n,1})$, then we have $x + y = n - 2$. In any other case we have $x + y > n - 2$. Therefore we have $x + y \geq n - 2$. □

Lemma 3.8. *Let f be an $L(h,k)$ labeling of $K_n \Box P_2$, where $\frac{nh}{2} < k \leq (n-1)h$, n is an even number greater than 4 and there are two transition between the layers. Then the highest label is greater than or equal to $2k + (n-2)h + (n-3)h$.*

Proof. Consider Figure 30.

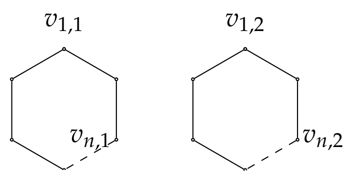

Figure 30: $K_n \Box P_3$

Let us assume that the vertex with the smallest label is in the second layer and let that vertex be $v_{1,2}$. Let $v_{1,1}$ be the vertex with the smallest label in the first layer. Let $v_{n,1}$ be the vertex with highest label in the first layer.

Let there be x vertices in the second layer that are at a distance of two from $v_{1,1}$ and have label lower than $f(v_{1,1})$.

Let there be y vertices in the second layer that are at a distance of two from $v_{n,1}$ and have label higher than $f(v_{n,1})$. From Lemma 3.7 we know that $x + y \geq n + 2$

Let us first assume that $x \geq 1$ and $y \geq 1$. We have $f(v_{1,1}) \geq (x-1)h + k$. It follows that $f(v_{n,1}) \geq (x-1)h + k + (n-1)h$.

Let $v_{n,2}$ be the vertex with highest label in the second layer. Now since there are y vertices in the second layer that are at a distance of two from $v_{n,1}$ and have labels greater than $f(v_{n,1})$, we have

$$
\begin{aligned}
f(v_{n,2}) &\geq (x-1)h + k + (n-1)h + k + (y-1)h \\
&= k + (n-1)h + k + (x+y-1)h \\
&\geq 2k + (n-1)h + (n-3)h
\end{aligned}
$$

Now when x is zero we have $y \geq n-2$. Since the vertex with smallest label $v_{1,2}$, is in the second layer, We have $(n-1)$ vertices that are at a distance of two from $v_{1,2}$. Therefore $f(v_{n,1}) \geq (n-2)h + k$. Therefore

$$
\begin{aligned}
f(v_{n,2}) &\geq (n-2)h + k + k + (y-1)h \\
&\geq 2k + (n-2)h + (n-3)h.
\end{aligned}
$$

When y is zero, we have $x \geq n-2$ and $f(v_{1,1}) \geq (x-1)h+k$. Hence we have $f(v_{n,1}) \geq (x-1)h+k+(n-1)h$. Since $y=0$, we have $d(v_{n,1},v_{n,2})=1$. Let $v_{n-1,1}$ be the vertex with second highest label in the first layer. We have $f(v_{n-1,1}) \geq (x-1)h+k+(n-2)h$. Also we know that $d(v_{n-1,1},v_{n,2})=2$. Hence we have

$$\begin{aligned} f(v_{n,2}) &\geq (x-1)h+k+(n-2)h+k \\ &\geq 2k+(n-2)h+(n-3)h. \end{aligned}$$

□

Lemma 3.9. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is an even number greater than 4. Let us arrange all $2n$ vertices in increasing order of their label. So $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_{i+1}) - f(v_i)$. If there exist x_j and x_l such that $x_j \geq k$ and $x_l \geq k$ then have $f(v_{2n}) \geq 3k+(n-2)h$.*

Proof. We have $2n-1$ x_i's. Since we know that any two vertices will be either at a distance of one or at a distance of two, we have $x_i \geq h$ for each i. So we have

$$\begin{aligned} x_1+x_2+x_3+\ldots+x_j+\ldots+x_l+\ldots+x_{2n-1} &\geq (2n-3)h+2k \\ &\geq 3k+(n-2)h(\text{ since } k \leq (n-1)h) \end{aligned}$$

□

Lemma 3.10. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is an even number greater than 4. Let us arrange all $2n$ vertices in increasing order of their label. So $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_{i+1}) - f(v_i)$. If there exist x_j, x_l, x_m such that $x_j + x_{j+1} \geq k$, $x_l + x_{l+1} \geq k$ and $x_m + x_{m+1} \geq k$, then $f(v_{2n}) \geq 3k+(n-2)h$.*

Proof. We have $2n-1$ x_i's. Since we know that any two vertices are either at a distance of one or two, we have $x_i \geq h$ for each i. So we have $x_1+x_2+x_3+\ldots+x_j+\ldots+x_l+\ldots+x_{2n-1} \geq 3k+(2n-7)h \geq 3k+(n-5)h+(n-2)h \geq 3k+(n-2)h$ for $n \geq 5$. □

Lemma 3.11. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is an even number greater than 4. Let us arrange all $2n$ vertices in increasing order of their labels. So $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_{i+1}) - f(v_i)$. If there exist x_j, x_l, x_m such that $x_j \geq k$, $x_l + x_{l+1} \geq k$ and $x_m + x_{m+1} \geq k$, then $f(v_{2n}) \geq 3k+(n-2)h$.*

Proof. We have $2n-1$ x_i's and since we know that any two vertices are either at a distance of one or two, we have $x_i \geq h$ for each i. So we have

$$\begin{aligned} x_1 + x_2 + x_3 + \ldots + x_j + \ldots + x_l + \ldots + x_{2n-1} &\geq 3k + (2n-6)h \\ &\geq 3k + (n-4)h + (n-2)h \\ &\geq 3k + (n-2)h \end{aligned}$$

□

Lemma 3.12. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is an even number greater than 4. Let us arrange all $2n$ vertices in increasing order of their labels. So $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_{i+1}) - f(v_i)$. If we have a combined two transition under f, then there exists an x_j such that $x_j \geq k$.*

Proof. Since we have a combined two transition there exists a set of vertices v_i, v_{i+1}, v_{i+2} such that v_i is in first layer (since we just have two layers we can start with any layer), v_{i+1} is in second layer and v_{i+2} is in first layer. Now the vertex v_{i+1} can either be adjacent to v_i or v_{i+2} but not both. So either $d(v_i, v_{i+1}) = 2$ or $d(v_{i+2}, v_{i+1}) = 2$. Therefore either $x_i \geq k$ or $x_{i+1} \geq k$. □

Lemma 3.13. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is an even number greater than 4. Let us arrange all the $2n$ vertices in the increasing order of their label. So $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_{i+1}) - f(v_i)$. If we have a combined three transition under f then there exist x_j and x_m such that $x_j \geq k$ and $x_m + x_{m+1} \geq k$.*

Proof. Since we have a combined three transition there exist a set of vertices v_i, v_{i+1}, v_{i+2}, v_{i+3} such that v_i is in first layer (since we just have two layers we can start with any layer), v_{i+1} is in second layer, v_{i+2} is in first layer and v_{i+3} is in second layer. The vertex v_{i+1} can either be adjacent to v_i or v_{i+2} but not both. Hence either $d(v_i, v_{i+1}) = 2$ or $d(v_{i+2}, v_{i+1}) = 2$. Therefore either $x_i \geq k$ or $x_{i+1} \geq k$.
If $x_i \geq k$, then consider the system of vertices $v_{i+1}, v_{i+2}, v_{i+3}$ we have a combined two transition. By Lemma 3.12 there exists an x_j such that $x_j \geq k$. Hence $x_j + x_{j+1} \geq k$
When $x_{i+1} \geq k$, if v_{i-1} exists it will be in first layer since we just have combined three transition therefore we will have $d(v_{i-1}, v_{i+1}) = 2$ hence $x_{i-1} + x_i \geq k$. If v_{i+4} exists it will be in second layer since we just have combined three transition therefore we will have $d(v_{i+4}, v_{i+2}) = 2$. Hence $x_{i+2} + x_{i+3} \geq k$. Since $n \geq 6$ either v_{i-1} or v_{i+4} should exist. □

Lemma 3.14. *Let f be an $L(h,k)$ labeling of K_n, where $\frac{nh}{2} < k \leq (n-1)h$, n is an even number greater than 4, and the highest label is less than $(\frac{n}{2}-1)h + k$. When the vertices of the graph are arranged* *in increasing order of their labeling, that is* $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n)$*, then* $f(v_{i+1} - f(v_i) < k - (\frac{n}{2} - 1)h$

Proof. Consider some vertex v_i such that $|f(v_{i+1}) - f(v_i)| \geq k - (\frac{n}{2} - 1)h$. We know that $f(v_i) \geq (i-1)h$. So $f(v_{i+1}) \geq (i-1)h + k - (\frac{n}{2} - 1)h$. Therefore $f(v_n) \geq (i-1)h + k - (\frac{n}{2} - 1)h + (n - i - 1)h \geq (\frac{n}{2} - 1)h + k$ which is a contradiction. □

Lemma 3.15. *Let f be an $L(h,k)$ labeling of $K_n \square P_3$, where $\frac{nh}{2} \leq k \leq (n-1)h$, n is an even number greater than 4, and zero is in first layer. If the difference between highest and lowest labels in both first and third layers is less than $(\frac{n}{2} - 1)h + k$ then in third layer lowest label is greater than or equal to k and highest label is greater than or equal to $(n-1)h + k$.*

Proof. Let us arrange the vertices of the first layer in increasing order of their labeling, that is, $f(v_{1,1}) < f(v_{2,1}) < f(v_{3,1}) < \ldots < f(v_{n-1,1}) < f(v_{n,1})$. Assume that there exists a vertex $v_{i,1}$ such that $f(v_{i,3}) \geq f(v_{i,1}) + k$ (we will surely have a vertex like this since zero is in first layer) and $f(v_{i+1,3}) \leq f(v_{i+1,1}) - k$, where $d(v_{i,1}, v_{i,3}) = 2$ and $d(v_{i+1,1}, v_{i+1,3}) = 2$. Now $|f(v_{i,3}) - f(v_{i+1,3})| \geq |2k - (f(v_{i+1,1}) - f(v_{i,1}))|$. From Lemma 3.14, we know that $(f(v_{i+1,1}) - f(v_{i,1})) \leq k - (\frac{n}{2} - 1)h$. Hence we have $|f(v_{i,3}) - f(v_{i+1,3})| \geq 2k - (k - (\frac{n}{2} - 1)h \geq (\frac{n}{2} - 1)h + k$, which is a contradiction. Therefore for every vertex $v_{i,1}$ in the first layer we have $f(v_{i,3}) \geq f(v_{i,1}) + k$. Since $f(v_{1,1}) \geq 0$, we have $f(v_{1,3}) \geq k$ and since $f(v_{n,1}) \geq (n-1)h$, $f(v_{n,3}) \geq (n-1)h + k$. □

Lemma 3.16. *If $\frac{nh}{2} < k \leq (n-1)h$ and n is even, then $\lambda(K_n \square P_2) \geq (n-1)h + (n-2) + k$.*

Proof. Let f be an optimal $L(h,k)$ labeling of $K_n \square P_2$. Now let us arrange all the $2n$ vertices of the graph in increasing order of their labels, that is, $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{2n-2}) < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_{i+1}) - f(v_i)$. There should be at least one transition in the graph so there exists x_l such that $x_l + x_{l+1} \geq k$. Hence

$$
\begin{aligned}
x_1 + x_2 + x_3 + \ldots + x_j + \ldots + x_l + \ldots + x_{2n-1} &\geq k + (2n-3)h \\
&\geq (n-1)h + (n-2)h + k
\end{aligned}
$$

□

Lemma 3.17. *Let f be an $L(h,k)$ labeling of $K_n \square P_3$, where $\frac{nh}{2} \leq k \leq (n-1)h$ and n is even. If the difference between highest and lowest labels in both first and third layers is less than $(\frac{n}{2} - 1)h + k$, then we have highest label greater than or equal to $(n-2)h + 3k$.*

Proof. Assume that zero is in first layer and since the difference between highest and lowest label in the first layer is less than $(\frac{n}{2}-1)h+k$, by Lemma 3.4 lowest label in second layer is greater than or equal to $(n-2)h+k$ and by Lemma 3.15 the lowest label in the third layer is greater than or equal to k. So every vertex in the second and third layers has label greater than k and by Lemma 3.16 we know that the difference between highest and lowest labels in second and third layers should be greater than $(n-1)h+k+(n-2)h$. Therefore the highest label for the graph is greater than or equal to $(n-1)h+k+(n-2)h+k \geq (n-2)h+3k$ (since $k \leq (n-1)h$).
If zero is in second layer, we can only have either one transition or two transitions between first and second layers(since we cannot use any label between $v_{i,1}$ and $v_{i+1,1}$ for the second layer by Lemma 3.4). If there is only one transition between first and second layers we have highest label in first layer greater than or equal to $(n-1)h+k+(n-2)h$. Therefore by Lemma 3.15 highest label in the third layer greater than or equal to $(n-1)h+k+(n-2)h+k \geq (n-2)h+3k$ (since $k \leq (n-1)h$).
If there are two transitions between first and second layers, by Lemma 3.8 we have the highest label in first layer greater than or equal to $(n-2)h+k$ and highest label in second layer is greater than or equal to $2k+(n-2)h+(n-3)h$. If there is only one transition between second and third layers and since zero is in second layer the highest label in third layer is greater than or equal to $2k+(n-1)h+(n-3)h+k+(n-2)h > 3k+(n-2)h$. Now if there are two transitions between second and third layers we have the highest label in second layer greater than or equal to $2k+(n-1)h+k+(n-3)h+(n-2)h \geq 3k+(n-2)h$.

□

Lemma 3.18. *Let f be an $L(h,k)$ labeling of $K_n \square P_2$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is an even number greater than 4. If there are more than three transitions under f, then we have highest label greater than or equal to $3k+(n-2)h$.*

Proof. Let us arrange the $2n$ vertices in increasing order of their labels. So we have $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_i) - f(v_{i+1})$. Therefore we have $2n-1$ x_i's. The different possible ways in which we can have four transitions are:

- Four isolated transitions.
- Two combined two transitions.
- One combined two transition and two isolated transitions.

- One combined three transition and one isolated transition.
- One combined four transition.

Case 1. *Four isolated transitions:*

Let us say we have four isolated transitions at following vertices v_i, v_j, v_l, v_m. Therefore we will have $x_i + x_{i+1} \geq k$, $x_j + x_{j+1} \geq k$, $x_l + x_{l+1} \geq k$, $x_m + x_{m+1} \geq k$. Hence by lemma 3.10 the highest label is greater than or equal to $3k + (n-2)h$.

Case 2. *Two combined two transitions.*

In a combined two transition we have an x_j such that $x_j \geq k$. Since we have two combined two transitions, we have $x_j \geq k$ and $x_l \geq k$. Hence by lemma 3.9 we have the highest label greater than $3k + (n-2)h$.

Case 3. *One combined two transition and two isolated transitions.*

In a combined two transition system we have an x_j such that $x_j \geq k$. In two isolated transitions we have x_l and x_m such that $x_l + x_{l+1} \geq k$ and $x_m + x_{m+1} \geq k$. Hence by lemma 3.11 we have highest label greater than $3k + (n-2)h$.

Case 4. *One combined three transition and one isolated transition.*

In a combined three transition system we have x_l and x_m such that $x_l \geq k$ and $x_m + x_{m+1} \geq k$ and in an isolated transition we have x_n such that $x_n + x_{n+1} \geq k$. Hence by lemma 3.11 we have highest label greater than $3k + (n-2)h$.

Case 5. *One combined four transition.*

Since we have four transitions we have a set of vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ such that v_i, v_{i+2}, v_{i+4} are in first layer. v_{i+1} and v_{i+3} are in second layer. Now either $d(v_i, v_{i+1}) = 2$ or $d(v_{i+1}, v_{i+2}) = 2$. Similarly either $d(v_{i+2}, v_{i+3}) = 2$ or $d(v_{i+4}, v_{i+4}) = 2$. So we have at least have two x_i such that $x_i \geq k$. Therefore by Lemma 3.9 we have highest label greater than $3k + (n-2)h$.

Hence by the above five cases we can say that highest label will be greater than or equal to $3k + (n-2)h$. Also if we have more than four transitions then the highest label will be greater than $3k + (n-2)h$. □

Lemma 3.19. *Let f be an $L(h,k)$ labeling of $K_n \square P_3$ where $\frac{nh}{2} < k \leq (n-1)h$ and n is even. If there are three transitions under f, then the highest label is greater than or equal to $3k + (n-2)h$.*

Proof. Let us arrange the $2n$ vertices in increasing order of their labels. So we have $f(v_1) < f(v_2) < f(v_3) < \ldots < f(v_{n-1}) < f(v_n) < \ldots < f(v_{2n-1}) < f(v_{2n})$. Let $x_i = f(v_i) - f(v_{i+1})$. Therefore we have $2n-1$ x_i's. The different possible ways in which we can have the transitions are:

- Three isolated transitions.
- One combined three transition.
- One combined two transition and one isolated transitions.

Case 1. *Three isolated transitions:*

Let us say we have isolated transitions at vertices v_i, v_j, v_l. Therefore we have $x_i + x_{i+1} \geq k$, $x_j + x_{j+1} \geq k$, $x_l + x_{l+1} \geq k$. Hence by Lemma 3.10 we have highest label greater than or equal to $3k + (n-2)h$.

Case 2. *One combined three transition:*

In a combined three transition we have a set of vertices v_i, v_{i+1}, v_{i+2}, v_{i+3} such that v_i is in first layer, v_{i+1} is in second layer,v_{i+2} is in first layer,v_{i+3} is in second layer.
If $d(v_{i+1}, v_{i+2}) = 1$ then $d(v_i, v_{i+1}) = 2$ and $d(v_{i+2}, v_{i+3}) = 2$. Therefore $x_i \geq k$ and $x_{i+2} \geq k$, so by Lemma 3.9 we highest label greater than or equal to $3k + (n-2)h$.
If $d(v_{i+1}, v_{i+2}) = 2$ then we can have a scenario such that $d(v_i, v_{i+1}) = 1$ and $d(v_{i+2}, v_{i+3}) = 1$. Therefore $x_{i+1} \geq k$. The vertex v_{i-1} should be in first layer(since we just have combined three transition system), also $d(v_{i-1}, v_{i+2}) = 2$, therefore $x_i + x_{i+1} \geq k$. The vertex v_{i+4} should be in second layer(since we just have combined three transition system), also $d(v_{i+2}, v_{i+4}) = 2$, therefore $x_{i+2} + x_{i+3} \geq k$. If v_{i-1} does not exist we will have an isolated transition after v_{i+3} because each layer has at least 6 vertices, similarly if v_{i+4} does not exist we will have an isolated transition before v_i. So we will have $x_j + x_{j+1} \geq k$ for some j. Hence by Lemma 3.11 we have highest label greater than or equal to $3k + (n-2)h$.

Case 3. *One combined two transition and one isolated transition.*

In a combined two transition system we have a set of vertices v_i, v_{i+1} and v_{i+2} such that v_i is in first layer. v_{i+1} is in second and v_{i+2} is in first. Now either v_{i-1} or v_{i+3} should exits.

If both exist, then we have a set of x_j and x_l such that $x_j \geq k$ and $x_l + x_{l+1} \geq k$. Also we have an isolated transition therefore we have x_m such that $x_m + x_{m+1} \geq k$. Hence by Lemma 3.11 we have the highest label greater than or equal to $3k + (n-2)h$.

If v_{i-1} does not exits and $d(v_{i+1}, v_{i+2}) = 2$. Then we just have $x_{i+1} \geq k$. Since we also have an isolated transition there exists an x_m such that $x_m + x_{m+1} \geq k$. This corresponds to the following special case:

The vertex v_1 is in second layer. v_2 is in first layer and v_3 is in second layer, then v_{n+1} is in second layer and from v_{n+2} to v_{2n} all the vertices are in first layer. The $d(v_2, v_3) = 2$. Since $f(v_2) \geq h$, we have $f(v_3) \geq h + k$. So $f(v_{n+1}) \geq h + k + (n-2)h$, $f(v_{n+2}) \geq (n-2)h + 2k$ and $f(v_{2n} \geq (n-2)h + 2k + (n-2)h$.

Since the third layer is adjacent to second layer. The vertex which is at a distance of two from v_{n+2} will have label $(n-2)h + 2k + k \geq 3k + (n-2)h$.

Hence we have the highest label greater than or equal to $3k + (n-2)h$. □

Lemma 3.20. *Let f be an $L(h,k)$ labeling of $K_n \Box P_4$, Where $\frac{nh}{2} < k \leq (n-1)h$ and n is even.If f has two transitions between two adjacent layers then there exists at least one vertex v such that $f(v) \geq (n-2)h + 3k$*

Proof. Let us consider the two adjacent layers that have two transitions. Figure31 shows the layers that have two transitions between them.

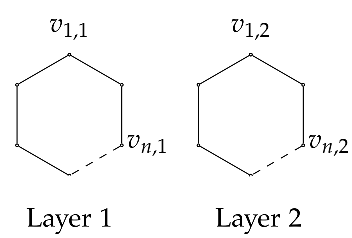

Figure 31: $K_n \Box P_3$

Without loss of generality assume that the vertex with the lowest label is in the second layer. Now arrange the vertices of first layer in the increasing order of their labels. Let the order be $f(v_{1,1}) < f(v_{2,1}) < f(v_{3,1}) < \ldots < f(v_{n-1,1}) < f(v_{n,1})$. Let there be x vertices in the second layer that have their labels less than $f(v_{1,1})$ and that are at a distance of two from $v_{1,1}$ and let there be y

vertices in the second layer that have their label greater than $f(v_{n,1})$ and which are at a distance of two from $v_{n,1}$.

Case 1. When $x \geq 1$ and $y \geq 1$.
We have $f(v_{1,1}) \geq (x-1)h + k$. We divide the problem into different sub cases depending on the difference between highest and lowest label in the first layer.
In the first sub-case assume that the difference between the highest and lowest label in the first layer is greater than or equal to $(\frac{n}{2}-1)h+k$. Therefore we have $f(v_{n,1}) \geq (x-1)h+k+(\frac{n}{2}-1)h+k$. We assumed that there are y vertices in the second layer which have labels greater than $f(v_{n,1})$ and which are at a distance of two from $v_{n,1}$. If $f(v_{n,2})$ is the vertex with the highest label in the second layer then $f(v_{n,2}) \geq f(v_{n,1}) + k + (y-1)h \geq (x-1)h + k + (\frac{n}{2}-1)h + k + k + (y-1)h = 3k + (x+y+\frac{n}{2}-3)h$. But we know that $x + y \geq n-2$ (from Lemma 3.7). Therefore we have $f(v_{n,2}) \geq 3k + (n-2)h + (\frac{n}{2}-3)h \geq 3k + (n-2)h$ for $n \geq 6$.
In the second sub-case assume that the difference between the highest and lowest label in the first layer is less than $(\frac{n}{2}-1)h+k$. Therefore by Lemma 3.8 we have $f(v_{n,2}) \geq 2k + (n-2)h + (n-3)h$. Suppose there is a layer adjacent to the layer 2 as shown in Figure 32.

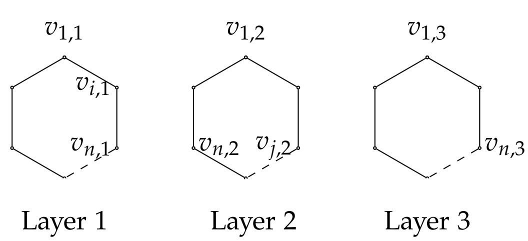

Figure 32: $K_n \square P_3$

We know that there are two transitions between layers 1 and 2 and layer 3 is adjacent to layer 2. From Lemma 3.17, if the difference between highest label in third layer is less than $(\frac{n}{2}-1)h+k$ then the highest label is greater than or equal to $3k+(n-2)h$. Hence the between the highest label and lowest label in third layer should be greater than or equal to $(\frac{n}{2}-1)h+k$. If there are two transitions between the second and third layers we have highest label greater than or equal to $3k+(n-2)h$, by the previous sub-case.
If there is a single transition between the second and third layers and if the vertex with the lowest label is in the second layer, then $f(v_{n,2}) \geq 2k+(n-2)h+(n-3)h$. We have the highest label

greater than or equal to $2k+(n-2)h+(n-3)h+(n-2)h > 3k+(n-2)h$ because there are $n-1$ vertices in the third layer that are at a distance of two from $v_{n,2}$.

Suppose the vertex with the lowest label is in the third layer. We know that the highest label in the third layer is greater than or equal to $(\frac{n}{2}-1)h+k$. Therefore we have $f(v_{1,2}) \geq (\frac{n}{2})h+k$, if it is at a distance of one from $v_{n,3}$. Now $f(v_{n,2}) \geq 2k+(n-2)h+(n-3)h+((\frac{n}{2})h+k) > 3k+(n-2)h$

Now if only layer one is adjacent to layer two as shown in Figure 33.

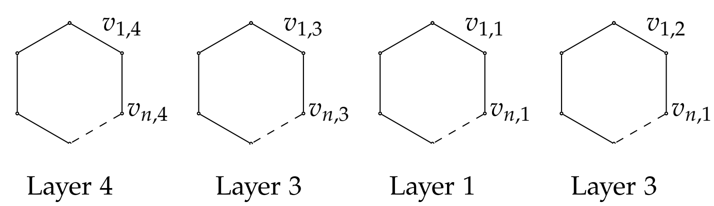

Figure 33: $K_n \square P_4$

If there are two transitions between Layer 3 and Layer 1 and if the vertex with smallest label is in layer 3, then we consider system of layers 1, 3 and 4. If layers 1 and 3 have two transitions with smallest labeled vertex in layer 3, since layer 4 is adjacent to layer three, then by previous sub-case of this lemma we have highest label greater than or equal to $3k+(n-2)h$.

If there are two transitions between layer and layer 1 and if the vertex with smallest label is in layer 1, then we consider system of layers 3, 1 and 2. If layers 1 and 3 have two transitions with smallest labeled vertex in layer 1 and layer 2 is adjacent to layer 1, then by previous part of this lemma we have highest label greater than or equal to $3k+(n-2)h$.

Now assume that there is only one transition between layer 3 and layer 1 and the vertex with the smallest label is in layer 3. Therefore we will have $f(v_{1,1}) \geq (n-2)h+k$. Hence we have $f(v_{n-1,1}) \geq (n-2)h+k+(n-2)h$ and $f(v_{n,1}) \geq (n-2)h+k(n-1)h$. Hence the vertex with highest label in second layer will have label $(n-2)h+k(n-1)h+k+(y-1)h \geq 3k+(n-2)h$ (since $k \leq (n-1)h$).

Now assume that the vertex having smallest label is in layer 1 and let that vertex be $v_{1,1}$. We know that $f(v_{1,1}) \geq (x-1)h+k$. Therefore $f(v_{n,1}) \geq (x-1)h+k+(n-1)h$. There are $(n-1)$ vertices in the third layer that are at a distance of two and whose labels are greater than $f(v_{n,1})$. Therefore we have highest label in the layer greater than or equal to $(x-1)h+k+(n-1)h+k+(n-2)h = 2k+(n-2)h+(n-1)h+(x-1)h \geq 3k+(n-2)h$ for $x \geq 1$ (since $k \leq (n-1)h$).

Case 2. when $x = 0$.

Consider the first and the second layers. There are two transitions between them. If $v_{2,1}$ is the lowest labeled vertex, then we have $d(v_{2,1}, v_{1,1}) = 1$, since $x = 0$.

Since $v_{2,1}$ is at a distance of two from $v_{1,2}$ we have $f(v_{2,1}) \geq k$. Therefore $f(v_{n,1}) \geq k + (n-2)h$.

If $v_{n,2}$ is the vertex with highest label in the second layer then we have $f(v_{n,2}) \geq k + (n-2)h + k + (y-1)h$ and also we know that $y \geq n-2$ from Lemma 3.7. Therefore we have $f(v_{n,2}) \geq k + (n-2)h + k + (n-3)h$.

Suppose there is a layer adjacent to the second layer as shown in Figure 34.

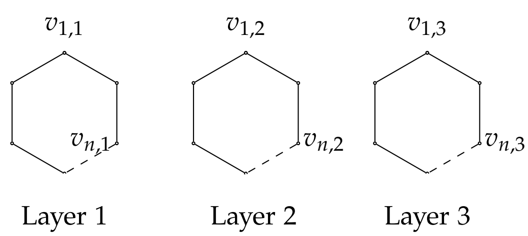

Figure 34: $K_n \square P_3$

We know that there are two transitions between layers 1, 2 and layer 3 is adjacent to layer two. From Lemma 3.17, if the difference between highest label and lowest label in third layer is less than $(\frac{n}{2} - 1)h + k$ we have our highest label greater than or equal to $3k + (n-2)h$. Hence we should have difference between the highest label in third layer greater than or equal to $(\frac{n}{2} - 1)h + k$. If there are two transitions between the second and third layers we have highest label greater than or equal to $3k + (n-2)h$, by the previous sub-case.

If there is a single transition between the second and third layers and if the vertex with the lowest label is in the second layer, then $f(v_{n,2}) \geq 2k + (n-2)h + (n-3)h$. We have the highest label greater than or equal to $2k + (n-2)h + (n-3)h + (n-2)h > 3k + (n-2)h$ because there are $n-1$ vertices in the third layer that are at a distance of two from $v_{n,2}$.

Suppose the vertex with the lowest label is in the third layer. We know that the highest label in the third layer is greater than or equal to $(\frac{n}{2} - 1)h + k$. Therefore we have $f(v_{1,2}) \geq (\frac{n}{2})h + k$, if it is at a distance of one from $v_{n,3}$. Now $f(v_{n,2}) \geq 2k + (n-2)h + (n-3)h + ((\frac{n}{2})h + k) > 3k + (n-2)h$.

Now if only layer 1 is adjacent to layer 2 as shown in Figure 35.

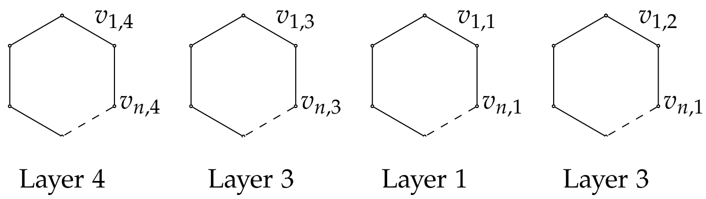

Figure 35: $K_n \square P_4$

If there are two transitions between layer 3 and layer 1 and if the vertex with smallest label in Layer 3, consider system of layers 1, 3, 4 . If layers 1 and 3 have two transitions with smallest labeled vertex in layer 3 and and layer 4 is adjacent to layer 3, then by previous sub-case of this lemma we have highest label greater than or equal to $3k + (n-2)h$.

If there are two transitions between layer three and layer one and if the vertex with smallest label is in layer 1, then consider system of layers 3, 1 and 2. If layers 1 and 3 have two transitions with smallest labeled vertex in layer 1 and layer two is adjacent to layer 1, then by previous part of this lemma we have highest label greater than or equal to $3k + (n-2)h$.

Now assume that there is only one transition between layer 3 and layer 1 and the vertex with the smallest label is in layer 3. Therefore we will have $f(v_{1,1}) \geq (n-2)h + k$. Hence we have $f(v_{n-1,1}) \geq (n-2)h + k + (n-2)h$ and $f(v_{n,1}) \geq (n-2)h + k(n-1)h$. Hence the vertex with highest label in second layer will have label $(n-2)h + k(n-1)h + k + (y-1)h \geq 3k + (n-2)h$ (since $k \leq (n-1)h$).

Now assume that the vertex having smallest label is in layer 1 and let that vertex be $v_{1,1}$. We know that $f(v_{1,1}) \geq (x-1)h + k$. Therefore $f(v_{n,1}) \geq (x-1)h + k + (n-1)h$. There are $(n-1)$ vertices in the third layer that are at a distance of two and whose labels are greater than $f(v_{n,1})$. Therefore we have highest label in the layer greater than or equal to $(x-1)h + k + (n-1)h + k + (n-2)h = 2k + (n-2)h + (n-1)h + (x-1)h \geq 3k + (n-2)h$ for $x \geq 1$ (since $k \leq (n-1)h$).

Case 3. When $y = 0$. Consider the first and the second layers. There are two transitions between these two layers. If $v_{1,1}$ is the lowest labeled vertex in the first layer $f(v_{1,1}) \geq (x-1)h + k$, $f(v_{n-1,1}) \geq (x-1)h + k + (n-2)h$ and $f(v_{n-1,1}) \geq (x-1)h + k + (n-1)h$. Therefore $f(v_{n,2}) \geq (x-1)h + k + (n-2)h + k$. This is same as result obtained in previous sub-case of this Lemma. Hence same proof follows.

$\square$

Corollary 3.21. *Let f be an $L(h,k)$ labeling of $K_n \square P_m$ with following conditions:*

- *There two or more than two transitions in some adjacent layers.*
- *n is even, $n \geq 6$ and $m \geq 5$.*
- *$\frac{nh}{2} < k \leq (n-1)h$.*
  Then $\lambda(K_n \square P_m) \geq (n-2)h + 3k$.

Proof. By Lemma's 3.18, 3.19, 3.20 we can say that if there are two or more than two transitions then the highest label is greater than or equal to $3k + (n-2)h$. ☐

Theorem 3.22. *Let f be an $L(h,k)$ labeling of $K_n \square P_m$ with following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *Zero is in the first layer.*
- *$\frac{nh}{2} < k \leq (n-1)h$.*
- *The difference between the highest label and lowest label in first layer is less than $(\frac{n}{2} - 1)h + k$.*

then $\lambda(K_n \square P_m) \geq (n-2)h + 3k$.

Proof. From Corollary 3.4, if there are two are more transitions then the highest label is greater than or equal to $3k + (n-2)h$. Let us consider the case when there is only one transition between any two adjacent layers. Consider three layers of K_n as shown in Figure 36.

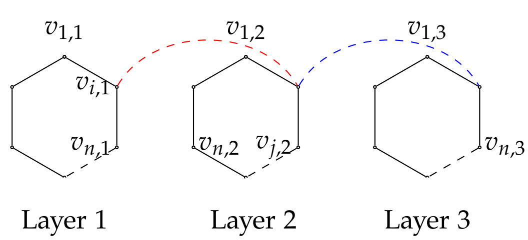

Figure 36: $K_n \square P_3$

Let $v_{1,i}$ be vertex with lowest label in the i^{th} layer. If $f(v_{1,1}) < f(v_{1,2}) < f(v_{1,3})$. Then by Lemma 3.7, we have the highest label greater than $(n-2)h + 3k$. Therefore if we take any three layers the possible configurations are:

$f(v_{1,1}) < f(v_{1,2})$ and $f(v_{1,3}) < f(v_{1,2})$
or $f(v_{1,1}) > f(v_{1,2})$ and $f(v_{1,3}) > f(v_{1,2})$.
Since zero is in first layer the only possible configuration for the five layers is as follows.
$f(v_{1,1}) < f(v_{1,2}), f(v_{1,3}) < f(v_{1,2}), f(v_{1,3}) < f(v_{1,4}), f(v_{1,5}) < f(v_{1,4})$.
So $f(v_{n,1}) < f(v_{n,2}), f(v_{n,3}) < f(v_{n,2}), f(v_{n,3}) < f(v_{n,4})$ and $f(v_{n,5}) < f(v_{n,4})$.
The highest label in the first layer is less than $(\frac{n}{2}-1)h+k$. By Lemma 3.17 the difference between the highest and lowest label cannot be less than $(\frac{n}{2}-1)h+k$. Also by Lemma 3.2, $f(v_{n-1,3}) \geq (\frac{n}{2}-1)h+k$ and $f(v_{n,3}) \geq (\frac{n}{2})h+k$.
Hence in second and fourth layers every vertex should have label greater than or equal to $(\frac{n}{2}-1)h+k+k$ and by Lemma 3.17 we know that in at least one layer we have the difference between highest and lowest label greater than or equal to $(\frac{n}{2}-1)h+k$. Let the vertex with highest label be $v_{n,4}$. So $f(v_{n,4}) \geq (\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h = 3k+(n-2)h$. □

Theorem 3.23. *Let f be an $L(h,k)$ labeling of $K_n \Box P_m$ with following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *Zero is in the first layer.*
- *$\frac{nh}{2} < k \leq (n-1)h$.*
- *The difference between the highest label and lowest label in first layer is greater than or equal to $(\frac{n}{2}-1)h+k$.*

then $\lambda(K_n \Box P_m) \geq (n-2)h+3k$ for $n > 3$.

Proof. From Corollary 3.4, if there are two are more transitions then the highest label is greater than or equal to $3k+(n-2)h$. Let us consider the case when there is only one transition between any two adjacent layers. Consider three layers of K_n as shown in Figure 37.

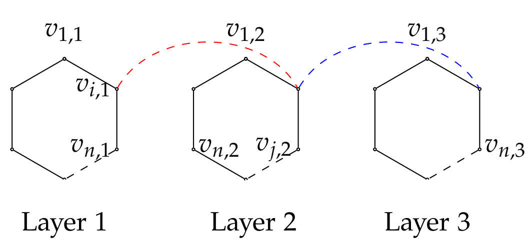

Figure 37: $K_n \Box P_3$

Let $v_{1,i}$ be vertex with lowest label in the i^{th} layer. If $f(v_{1,1}) < f(v_{1,2}) < f(v_{1,3})$. Then by Lemma 3.7, we have the highest label greater than $(n-2)h+3k$. Therefore if we take any three layers the possible configurations are:
$f(v_{1,1}) < f(v_{1,2})$ and $f(v_{1,3}) < f(v_{1,2})$
or $f(v_{1,1}) > f(v_{1,2})$ and $f(v_{1,3}) > f(v_{1,2})$.
Since zero is in first layer the only possible configuration for the five layers is as follows.
$f(v_{1,1}) < f(v_{1,2})$, $f(v_{1,3}) < f(v_{1,2})$, $f(v_{1,3}) < f(v_{1,4})$, $f(v_{1,5}) < f(v_{1,4})$.
So $f(v_{n,1}) < f(v_{n,2})$, $f(v_{n,3}) < f(v_{n,2})$, $f(v_{n,3}) < f(v_{n,4})$ and $f(v_{n,5}) < f(v_{n,4})$
Now consider the third layer and let $v_{n,3}$ be the vertex with highest label. If $f(v_{n,3}) < (\frac{n}{2}-1)h+k$. Then by Lemma 3.5, the highest label in first and fifth layers is greater than $(\frac{n}{2})h+k$ and second highest label greater than $(\frac{n}{2}-1)h+k$. Therefore in second and fourth layers we have $f(v_{1,2}) \geq (\frac{n}{2}-1)h+k+k$ and $f(v_{1,4}) \geq (\frac{n}{2}-1)h+k+k$. Also by lemma 3.17 either in layer two and layer four the difference between highest and lowest label greater than or equal to $(\frac{n}{2}-1)h+k$. Therefore the highest label in at least one layer is greater than or equal to $(\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h+k = 3k+(n-2)h$.
Now consider the third layer and let $v_{n,3}$ be the vertex with highest label. If $f(v_{n,3}) \geq (\frac{n}{2}-1)h+k$. Let $v_{n-1,3}$ is the vertex with second highest label then we have $f(v_{n-1,3}) \geq (\frac{n}{2}-2)h+k$.
Now consider the second and fourth layers. We know that $f(v_{1,2}) \geq (\frac{n}{2}-2)h+k+k$ and $f(v_{1,4}) \geq (\frac{n}{2}-2)h+k+k$. Now if $f(v_{1,2}) < (\frac{n}{2}-1)h+k+k$. We have $|f(v_{n,3}) - f(v_{1,2})| < k$. Therefore they should be at a distance of one.
Now in the fourth layer if $f(v_{1,4}) < (\frac{n}{2}-1)h+k+k$. Then $v_{1,4}$ cannot be at a distance of two from $v_{n,3}$, because then we have $|f(v_{1,4}) - f(v_{n,3})| < k$. If the vertex $v_{1,4}$ is adjacent to $v_{n,3}$, then $d(v_{1,4}, v_{1,2}) = 2$ and $|f(v_{1,4}) - f(v_{1,2})| < k$. Hence we cannot have $f(v_{1,4}) < (\frac{n}{2}-1)h+k+k$. Therefore $f(v_{1,4}) \geq (\frac{n}{2}-1)h+k+k$. If $v_{n,4}$ is the vertex with highest label in the fourth layer then we have $f(v_{n,4}) \geq (\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h+k = 3k+(n-2)h$. □

Theorem 3.24. *Let f be an $L(h,k)$ labeling of $K_n \square P_m$ with following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *zero is in the second layer.*
- *$\frac{nh}{2} \leq k \leq (n-1)h$.*
- *The difference between the highest label and lowest label in second layer is less than $(\frac{n}{2}-1)h+k$.*

then $\lambda(K_n \square P_m) \geq (n-2)h+3k$ for $n > 3$.

Proof. From Corollary 3.4, if there are two are more transitions then the highest label is greater than or equal to $3k+(n-2)h$. Let us consider the case when there is only one transition between any two adjacent layers. Consider three layers of K_n as shown in Figure 38.

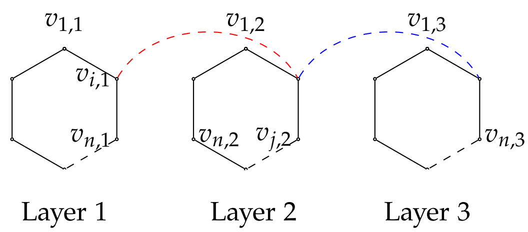

Figure 38: $K_n \square P_3$

If $f(v_{1,1}) < f(v_{1,2}) < f(v_{1,3})$. Then by Lemma 3.7, we have the highest label greater than $(n-2)h+3k$. Therefore if we take any three layers the possible configurations are:
$f(v_{1,1}) < f(v_{1,2})$ and $f(v_{1,3}) < f(v_{1,2})$
or $f(v_{1,1}) > f(v_{1,2})$ and $f(v_{1,3}) > f(v_{1,2})$.
Since zero is the second layer the only possible configuration of the five layers is:
$f(v_{1,1}) > f(v_{1,2})$, $f(v_{1,3}) > f(v_{1,2})$, $f(v_{1,3}) > f(v_{1,4})$ and $f(v_{1,5}) > f(v_{1,4})$.
The highest label in second layer is less than $(\frac{n}{2}-1)h+k$. So if $v_{n,4}$ and $v_{n-1,4}$ are the highest and second highest labeled vertices in the fourth layer respectively, by Lemma 3.5 we have $f(v_{n,4}) \geq (\frac{n}{2})h+k$ and $f(v_{n-1,4}) \geq (\frac{n}{2}-1)h+k$. In third and fifth layers at least in one layer the difference between the highest and lowest labels greater than or equal to $(\frac{n}{2}-1)h+k$ by Lemma 3.5, without loss of generality assume that this happens in fifth layer. If $v_{1,5}$ is the vertex with lowest label in the fifth layer and $v_{n,5}$ is the vertex with highest label in the fifth layer then $f(v_{1,5}) \geq (\frac{n}{2}-1)h+k+k$ and $f(v_{n,5}) \geq (\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h+k = 3k+(n-2)h$ $\square$

Theorem 3.25. *Let f be an $L(h,k)$ labeling of $K_n \square P_m$ with the following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *zero is in the second layer.*
- $\frac{nh}{2} \leq k \leq (n-1)h$.
- *The difference between the highest label and lowest label in second layer is greater than or equal to $(\frac{n}{2}-1)h+k$.*

then $\lambda(K_n \square P_m) \geq (n-2)h + 3k$.

Proof. From Corollary 3.4, if there are two are more transitions then the highest label is greater than or equal to $3k + (n-2)h$. Let us consider the case when there is only one transition between any two adjacent layers. Consider three layers of K_n as shown in Figure 39.

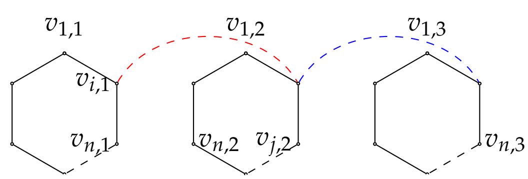

Figure 39: $K_n \square P_3$

If $f(v_{1,1}) < f(v_{1,2}) < f(v_{1,3})$. Then by Lemma 3.7, we have the highest label greater than $(n-2)h + 3k$. Therefore if we take any three layers the possible configurations are:
$f(v_{1,1}) < f(v_{1,2})$ and $f(v_{1,3}) < f(v_{1,2})$
or $f(v_{1,1}) > f(v_{1,2})$ and $f(v_{1,3}) > f(v_{1,2})$.
Since zero is the second layer the only possible configuration of the five layers is:
$f(v_{1,1}) > f(v_{1,2})$, $f(v_{1,3}) > f(v_{1,2})$, $f(v_{1,3}) > f(v_{1,4})$ and $f(v_{1,5}) > f(v_{1,4})$.
Consider fourth layer and let $v_{n,4}$ be the vertex with the highest label. If $f(v_{n,4}) < (\frac{n}{2} - 1)h + k$. Then $f(v_{n,2}) \geq (\frac{n}{2})h + k$ and $f(v_{n-1,2}) \geq (\frac{n}{2} - 1)h + k$. In first and third layer at least in one layer the difference between the highest and lowest labels is greater than or equal to $(\frac{n}{2} - 1)h + k$ by Lemma 3.17. Assume that we have the above criteria in the third layer, if $v_{1,3}$ is the vertex with lowest label in the third layer and $v_{n,3}$ is the vertex with highest label in the third layer then $f(v_{1,3}) \geq (\frac{n}{2} - 1)h + k + k$ and $f(v_{n,3}) \geq (\frac{n}{2} - 1)h + k + k + (\frac{n}{2} - 1)h + k = 3k + (n-2)h$.
If the difference between the highest and lowest labels in fourth layer is greater than or equal to $(\frac{n}{2} - 1)h + k$, then $f(v_{n,4}) \geq (\frac{n}{2} - 1)h + k$ and $f(v_{n-1,4}) \geq (\frac{n}{2} - 2)h + k$.
If the difference between highest and lowest label in the third layer is less than $(\frac{n}{2} - 1)h + k$, then in fifth layer $|f(v_{n,5}) - f(v_{1,5})| \geq (\frac{n}{2})h + k$. Since $f(v_{1,5}) \geq (\frac{n}{2} - 2)h + k + k$ we have $f(v_{n,5}) \geq (\frac{n}{2} - 2)h + k + k + (\frac{n}{2})h + k = 3k + (n-2)h$.
If the difference between lowest and highest label in both third layer and fifth layers is greater

than or equal to $(\frac{n}{2}-1)h+k$ then $f(v_{1,3}) \geq (\frac{n}{2}-2)h+k+k$ and $f(v_{1,5}) \geq (\frac{n}{2}-2)h+k+k$. If $f(v_{1,3}) < (\frac{n}{2}-1)h+k+k$ then $|f(v_{n,4})-f(v_{1,3})| < k$, therefore they are at a distance of one.
If in fifth layer $f(v_{1,5}) < (\frac{n}{2}-1)h+k+k$, then $v_{1,5}$ cannot be at a distance of two from $v_{n,3}$, since $|f(v_{1,5})-f(v_{n,4})| < k$. If the vertex $v_{1,5}$ is adjacent to $v_{n,3}$ then $d(v_{1,5},v_{1,3}) = 2$ and $|f(v_{1,4})-f(v_{1,2})| < k$. Hence $f(v_{1,4})$ cannot be less than $(\frac{n}{2}-1)h+k+k$. Therefore $f(v_{1,4}) \geq (\frac{n}{2}-1)h+k+k$ and hence $f(v_{n,4}) \geq (\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h+k = 3k+(n-2)h$. $\square$

Theorem 3.26. *Let f be an $L(h,k)$ labeling of $K_n \square P_m$ with following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *zero is in the third layer.*
- *$\frac{nh}{2} < k \leq (n-1)h$.*
- *The difference between the highest label and lowest label in the third layer is less than $(\frac{n}{2}-1)h+k$.*

then $\lambda(K_n \square P_m) \geq (n-2)h+3k$ for $n > 3$.

Proof. From Corollary 3.4, if there are two are more transitions then the highest label is greater than or equal to $3k+(n-2)h$. Let us consider the case when there is only one transition between any two adjacent layers. Consider three layers of K_n as shown in Figure 40.

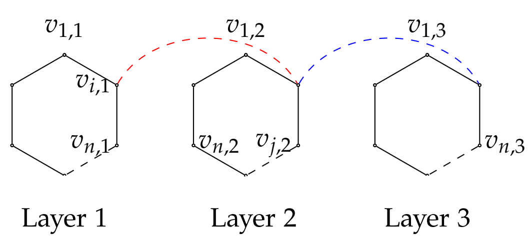

Figure 40: $K_n \square P_3$

Let $v_{1,i}$ be vertex with lowest label in the i^{th} layer. If $f(v_{1,1}) < f(v_{1,2}) < f(v_{1,3})$. Then by Lemma 3.7, we have the highest label greater than $(n-2)h+3k$. Therefore if we take any three layers the possible configurations are:
$f(v_{1,1}) < f(v_{1,2})$ and $f(v_{1,3}) < f(v_{1,2})$
or $f(v_{1,1}) > f(v_{1,2})$ and $f(v_{1,3}) > f(v_{1,2})$.

Since zero is in third layer the only possible configuration for the five layers is as follows.
$f(v_{1,1}) < f(v_{1,2}), f(v_{1,3}) < f(v_{1,2}), f(v_{1,3}) < f(v_{1,4}), f(v_{1,5}) < f(v_{1,4})$.
So $f(v_{n,1}) < f(v_{n,2}), f(v_{n,3}) < f(v_{n,2}), f(v_{n,3}) < f(v_{n,4})$ and $f(v_{n,5}) < f(v_{n,4})$
Consider the third layer and let $v_{n,3}$ be the vertex with highest label. We know that $f(v_{n,3}) < (\frac{n}{2}-1)h+k$. Therefore in first and fifth layers we have highest label greater than $(\frac{n}{2})h+k$ and second highest label greater than $(\frac{n}{2}-1)h+k$ by Lemma 3.5. So $f(v_{1,2}) \geq (\frac{n}{2}-1)h+k+k$ and $f(v_{1,4}) \geq (\frac{n}{2}-1)h+k+k$. By Lemma 3.3 at least in one among layers two and four the difference between highest and lowest label greater than or equal to $(\frac{n}{2}-1)h+k$. Therefore the highest label in at least one layer is greater than or equal to $(\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h+k = 3k+(n-2)h$. $\square$

Theorem 3.27. *Let f be an $L(h,k)$ labeling of $K_n \square P_5$ with following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *zero is in the third layer.*
- *$\frac{nh}{2} < k \leq (n-1)h$.*
- *The difference between the highest label and lowest label in the third layer is greater than or equal to $(\frac{n}{2}-1)h+k$.*

then $\lambda(K_n \square P_5) \geq (n-2)h+3k$.

Proof. From Corollary 3.4, if there are two are more transitions then the highest label is greater than or equal to $3k+(n-2)h$. Let us consider the case when there is only one transition between any two adjacent layers. Consider three layers of K_n as shown in Figure 41.

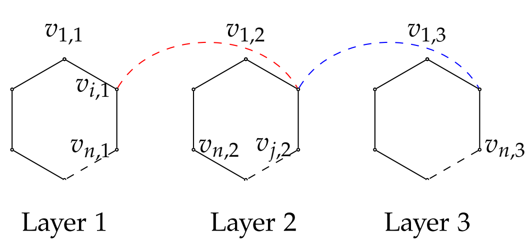

Figure 41: $K_n \square P_3$

If $f(v_{1,1}) < f(v_{1,2}) < f(v_{1,3})$. Then by Lemma 3.6, we have the highest label greater than

$(n-2)h+3k$. Therefore if we take any three layers the possible configurations are:
$f(v_{1,1}) < f(v_{1,2})$ and $f(v_{1,3}) < f(v_{1,2})$
or $f(v_{1,1}) > f(v_{1,2})$ and $f(v_{1,3}) > f(v_{1,2})$.
Since zero is the second layer the only possible configuration of the five layers is:
$f(v_{1,1}) > f(v_{1,2})$, $f(v_{1,3}) > f(v_{1,2})$, $f(v_{1,3}) > f(v_{1,4})$ and $f(v_{1,5}) > f(v_{1,4})$.
In third layer $f(v_{n,3}) \geq (\frac{n}{2}-1)h+k$ and $f(v_{n-1,3}) \geq (\frac{n}{2}-2)h+k$.
We know that $f(v_{1,2}) \geq (\frac{n}{2}-2)h+k+k$ and $f(v_{1,4}) \geq (\frac{n}{2}-2)h+k+k$. Now if $f(v_{1,2}) < (\frac{n}{2}-1)h+k+k$ then $|f(v_{n,3})-f(v_{1,2})| < k$. Therefore they should be at a distance of one.
Now if $f(v_{1,4}) < (\frac{n}{2}-1)h+k+k$ then $v_{1,4}$ cannot be at a distance of two from $v_{n,3}$, because then $|f(v_{1,4})-f(v_{n,3})| < k$. If $v_{1,4}$ is adjacent to $v_{n,3}$, then $d(v_{1,4},v_{1,2}) = 2$ and $|f(v_{1,4})-f(v_{1,2})| < k$. Hence we cannot have $f(v_{1,4}) < (\frac{n}{2}-1)h+k+k$. So $f(v_{1,4}) \geq (\frac{n}{2}-1)h+k+k$. If $v_{n,4}$ is the vertex with highest label in the fourth layer then we have $f(v_{n,4}) \geq (\frac{n}{2}-1)h+k+k+(\frac{n}{2}-1)h+k = 3k+(n-2)h$. □

Theorem 3.28. *Let f be an $L(h,k)$ labeling of $K_n \Box P_m$ with the following conditions:*

- *n is even and $n \geq 6$, $m \geq 5$.*
- *$\frac{nh}{2} < k \leq (n-1)h$. then $\lambda(K_n \Box P_m) = (n-2)h+3k$.*

Proof. From Corollary 3.4, if there are two or more than two transitions between adjacent layers then highest label is greater than or equal to $3k+(n-2)h$ and from Theorems 3.22, 3.23, 3.24, 3.25, 3.26, 3.27 we can say that $\lambda_{K_n \Box P_5} \geq 3k+(n-2)h$. Since $K_n \Box P_5$ is an induced subgraph of $K_n \Box P_m$, we have $\lambda_(K_n \Box P_m) \geq (n-2)h+k$. Also we have a labeling with highest label as $(n-2)h+3k$.The labeling is as follows
For the j layer:
when $j \equiv 1 \pmod 4$,for any vertex $v_{i,j}$, we have

$$f(v_{i,j}) = \begin{cases} (i-1)h, & \text{for } 1 \leq i \leq \frac{n}{2} \\ (i-1-\frac{n}{2})h+k, & \text{for } \frac{n}{2} < i \leq n \end{cases}$$

when $j \equiv 2 \pmod 4$,for any vertex $v_{i,j}$, we have

$$f(v_{i,j}) = \begin{cases} (\frac{n}{2}-1)h+2k+(i-1)h, & \text{for } 1 \leq i \leq \frac{n}{2} \\ (\frac{n}{2}-1)h+3k+(i-1-\frac{n}{2})h, & \text{for } \frac{n}{2} < i \leq n \end{cases}$$

when $j \equiv 3 \pmod 4$,for any vertex $v_{i,j}$,

$$f(v_{i,j}) = \begin{cases} (i-1)h+k, & \text{for } 1 \le i \le \frac{n}{2} \\ (i-1-\frac{n}{2})h, & \text{for } \frac{n}{2} < i \le n \end{cases}$$

when $j \equiv 0 \pmod 4$,for any vertex $v_{i,j}$,

$$f(v_{i,j}) = \begin{cases} (\frac{n}{2}-1)h+3k+(i-1)h, & \text{for } 1 \le i \le \frac{n}{2} \\ (\frac{n}{2}-1)h+2k+(i-1-\frac{n}{2})h, & \text{for } \frac{n}{2} < i \le n \end{cases}$$

We now give the proof of labeling:

Case 1. If the distance between any two vertices is one.

Consider two vertices v_i, v_j in first layer.

If $f(v_{i,1}) = (i-1)h$ and $f(v_{j,1}) = (j-1)h$. Then

$$\begin{aligned} |f(v_{i,1}) - f(v_{j,1})| &= |(i-j)h| \\ &\ge h \end{aligned}$$

If $f(v_{i,1}) = (i-1)h$ and $f(v_{j,1}) = (j-1-\frac{n}{2})h+k$. Then

$$\begin{aligned} |f(v_{j,1}) - f(v_{i,1})| &= |(j-i-\frac{n}{2})h+k| \\ &\ge h \end{aligned}$$

If $f(v_{i,1}) = (i-1-\frac{n}{2})h+k$ and $f(v_{j,1}) = (j-1-\frac{n}{2})h+k$. Then

$$\begin{aligned} |f(v_{i,1}) - f(v_{j,1})| &= |(i-j)h| \\ &\ge h \end{aligned}$$

. Consider one vertex in first layer and one vertex in second layer:

If $f(v_{i,1}) = (i-1)h$ and $f(v_{i,2}) = (\frac{n}{2}-1)h+2k+(i-1)h$. Then

$$\begin{aligned} |f(v_{i,1}) - f(v_{i,2})| &= |(\frac{n}{2}-1)h+k+k| \\ &> h. \end{aligned}$$

The proof goes in similar fashion for other vertices which are at a distance of one.

Case 2. If the distance between any two vertices is two. Consider one vertex in first layer and one vertex in second layer.

If $f(v_{i_1,1}) = (i_1-1)h$ and $f(v_{i_2,2}) = (\frac{n}{2}-1)h+2k+(i_2-1)h$. Then

$$\begin{aligned} |f(v_{i_2,2}) - f(v_{i_1,1})| &= |(\frac{n}{2}-1)h+2k+(i_2-i_1)h| \\ &> k. \text{ (Since } (i_2-i_1)h < (\frac{n}{2}-1)h+k) \end{aligned}$$

If $f(v_{i_1,1}) = (i_1-1)h$ and $f(v_{i_2,2}) = (\frac{n}{2}-1)h+3k+(i_2-1-\frac{n}{2})h$. Then

$$\begin{aligned}|f(v_{i_2,2}) - f(v_{i_1,1})| &= |(\tfrac{n}{2}-1)h+3k+(i_2-i_1)h| \\ &> k \text{ (Since } (i_2-i_1)h < (\tfrac{n}{2}-1)h+k)\end{aligned}$$

If $f(v_{i_1,1}) = (i_1-1-\frac{n}{2})h+k$ and $f(v_{i_2,2}) = (\frac{n}{2}-1)h+2k+(i_2-1)h$. Then

$$\begin{aligned}|f(v_{i_2,2}) - f(v_{i_1,1})| &= |(\tfrac{n}{2}-1)h+2k+(i_2-i_1)h+(\tfrac{n}{2})h-k| \\ &= |(n-1)h+k-(i_1-i_2)h| \\ &> k \text{ (Since } (i_2-i_1)h < (n-1)h\end{aligned}$$

If $f(v_{i_1,1}) = (i_1-1-\frac{n}{2})h+k$ and $f(v_{i_2,2}) = (\frac{n}{2}-1)h+3k+(i_2-1-\frac{n}{2})h$. Then

$$\begin{aligned}|f(v_{i_2,2}) - f(v_{i_1,1})| &= |2k+(i_2-i_1)h|+(\tfrac{n}{2}-1)h \\ &> k\end{aligned}$$

If one vertex is in first layer and one vertex is in third layer.
If $f(v_{i,1}) = (i-1)h$ and $f(v_{i,3}) = (i-1)h+k$. Then

$$\begin{aligned}|f(v_{i,3}) - f(v_{i,1})| &= |(i-1)h+k-(i-1)h| \\ &> k\end{aligned}$$

If $f(v_{i,1}) = (i-1-\frac{n}{2})h+k$ and $f(v_{i,3}) = (i-1-\frac{n}{2})h$. Then

$$\begin{aligned}|f(v_{i,3}) - f(v_{i,1})| &= |(i-1-\tfrac{n}{2})h+k-(i-1-\tfrac{n}{2})h| \\ &> k\end{aligned}$$

The proof goes in similar fashion for other vertices which are at a distance of one.

□

In Figure 42 we show $L(5,21)$ labeling of $K_6 \square P_5$

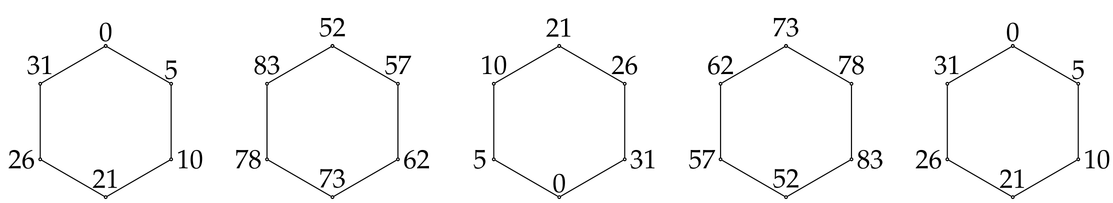

Figure 42: $L(5,21)$ labeling for $K_6 \square P_5$

4. Lower bounds for other graphs

Since $K_n \square P_{m-1}$ is an induced subgraph of $K_n \square C_m$, from Lemma 3.1 we can get the lower bound for $K_n \square C_m$.

Corollary 4.1. *If $h \geq 2k$ and $m > 2$ then $\lambda(K_n \square C_m) \geq (n-1)h + k$.*

Proof. From Theorem 3.1 we have $\lambda(K_n \square P_{m-1}) = (n-1)h + k$. Since $K_n \square P_{m-1}$ is an induced subgraph of $K_n \square C_m$, we have $\lambda(K_n \square C_m) \geq (n-1)h + k$. $\square$

Corollary 4.2. *If $2k \geq h \geq k$ and $m \geq 3$ then $\lambda(K_n \square C_m) \geq (2n-1)k$.*

Proof. From Theorem 3.2 we have $\lambda(K_n \square P_2) = (2n-1)k$. Since $K_n \square P_2$ is an induced subgraph of $K_n \square C_m$, we have $\lambda(K_n \square C_m) \geq (2n-1)k$. $\square$

Corollary 4.3. *If $h < k \leq \frac{nh}{2}$ then $\lambda(K_n \square C_m) \geq 2(n-1)h + k$ for $n \geq 3$ and $m \geq 4$.*

Proof. From Theorem 3.3 we have $\lambda(K_n \square P_3) = 2(n-1)h + k$. Since $K_n \square P_3$ is an induced subgraph of $K_n \square C_m$, we have $\lambda(K_n \square C_m) \geq 2(n-1)h + k$. $\square$

Corollary 4.4. *Consider $K_n \square C_m$ with the following conditions:*

- *n is even and $n \geq 6$ and $m \geq 6$.*
- *$\frac{nh}{2} < k \leq (n-1)h$.*

then $\lambda(K_n \square C_m) \geq (n-2)h + 3k$.

Proof. From Theorem 3.28 we have $\lambda(K_n \square P_m) = (n-2)h + 3k$. Since $K_n \square P_{m-1}$ is an induced subgraph of $K_n \square C_m$, we have $\lambda(K_n \square C_m) \geq (n-2)h + 3k$. $\square$

5. Future work

In the thesis $L(h,k)$ labeling is found when $k \leq (n-1)h$. We also determined the value of $\lambda(K_n \square P_m)$, when $k \geq nh$ and when n is odd by checking λ values for various graphs. We proved that when n is even and $k \geq (\frac{n}{2})h$ the optimal $L(h,k)$ labeling has only one transition between two adjacent layers. Using MATLAB simulations we checked that this is true when $k > (n-1)h$. From the above results we can derive the following conjectures.

Conjecture 5.1. *In the optimal $L(h,k)$ labeling of $K_n \square P_m$, there is only one transition between two adjacent layers when $k > h$.*

Conjecture 5.2. *In the optimal $L(h,k)$ labeling of $K_n \square C_m$, there is only one transition between two adjacent layers.*

Conjecture 5.3. *If $(\lfloor \frac{n}{2} \rfloor + 1)h < k \leq (n-1)h$ where n is odd then $\lambda(K_n \square P_m) = (n-1)h + 3k$.*

Conjecture 5.4. *If $k > (n-1)h$ then $\lambda(K_n \square P_m) = (3n-5)h + 2k$.*

6. Conclusion

In this thesis we completely solved the $L(h,k)$ labeling problem for $K_n \square P_m$ when $h \geq k$. The problem when $h < k$ is still a hypothetical concept for now. There are some papers which give $L(h,k)$ labelings for cases when $h < k$. Since it is a NP-hard problem, solving for cases when $h < k$ could give us important insights about the problem in general. If the Conjecture 5.1 is valid for any Cartesian product graphs then we can find $L(h,k)$ labeling of the graphs using polynomial time algorithms which takes us one step closer to solving $L(h,k)$ labeling problem for a general graph.

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